
Math 475

Combinatorics and Graph Theory

HW Set #3

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Key to Symbols and Notation

WLOG = Without Loss of Generality
${}_nP_r = P_r^n = P(n, r) =$ Number of Ordered Ways of Choosing r things from a set of n things.
${}_nC_r = C_r^n = C(n, r) = \binom{n}{r} =$ Binomial[n, r] = Choosing r things from a set of n things.
$n! = n$ Factorial
$[x_1\ x_2\ \ldots\ x_n]$ = 1-line notation for the permutation of n elements.
$(x_1\ x_2\ \ldots\ x_m)$ = a permutation cycle of m elements.
$< x_1, x_2, \ldots x_k >$ = multiset of k elements.
$\{x_1, x_2, \ldots x_l\}$ = set of l elements.

Problem #1

Chapter 4, #43

$X = \{a, b, c, d, e, f\}$ and R is a relation such that $a R b,\ b R c,\ c R d,\ a R e,\ e R f,\ f R d.$

Rather than trying to state in words why this is a covering relation, I'll just say that if you start with a minimal element and place everything it relates to on the next level, and then the things those relate to on the level above, etc. then a map is covering map iff each connection goes up exactly one level.

Possible linear extensions consist of choosing ways to order b, c, e, f between a and d, such that $e < f$ and $b < c$. Possible combinations are $a < b < c < e < f < d$; $a < b < e < f < c < d$; $a < b < e < c < f < d$; $a < e < f < b < c < d$; $a < e < b < f < c < d$; $a < e < b < c < f < d$.

Thus there are 6 possible linear extensions of this partial ordering.

Problem #2

How many *k*-letter words with exactly 5 vowels and no two vowels together are there in our 26 letter alphabet?

Firstly, we can treat this exactly as the case of choosing sticks without getting two sticks adjacent to each other. Thus we have k-letters and we wish to choose 5 vowels such that the letters are partitioned into 6 groups and all groups except the first and last are non-empty. If we denote the groups by $x_0, x_1, \ldots, x_5$ then $x_0, x_5 \geq 0$ and $x_1, x_2, \ldots x_4 \geq 1$ and $\sum_{i=0}^{5} x_i = k - 5$. Subsituting $y_0 = x_0$, $y_5 = x_5$ and $y_i = x_i - 1$, $\forall\ i \in \{1, 2, 3, 4\}$, we can rewrite the conditons as $y_i \geq 0$, $\forall\ i$ and $\sum_{i=0}^{5} y_i = k - 5 - (5 - 1)$.

Thus we just need the number of ways to partition the $k - 9$ things into the y_i's. Which is $\binom{k-9+5-1}{k-9} = \binom{k-5}{k-9} = \binom{k-5}{4}$. As a simple check we can note that this is 0 is $k < 9$, which is as it should since you can't possible have 5 vowels no two adjacent in a word of less than 9 letters.

Now to account for the distinct words we need only obeserve that $\exists$ 5 ways to pick each vowel and 21 ways to pick a consonant and each pick is totally independant. Thus $\exists \binom{k-5}{4} * 2^5 * 21^{k-5}$ distinct *k*-letter words with 5 vowels, no two of which adjacent.

Problem #3

Perfect Shuffles

1) Top Half of Deck in Left Hand

a) The cyclic notation for these permutations of the deck of 52 cards

The 1-line notation looks as follows [1 27 2 28 3 29 ... 26 52], thus n odd goes to $\frac{n+1}{2}$, and n even goes to $26 + \frac{n}{2}$. 1 and 52 will be fixed and the rest of the cycles are (2 27 14 33 17 9 5 3), (4 28 40 46 49 25 13 7), (6 29 15 8 30 41 21 11), (10 31 16 34 43 22 37 19), (12 32 42 47 24 38 45 23), (18 35), (20 36 44 48 50 51 26 39).

Thus this shuffle may be written as
(2 27 14 33 17 9 5 3) (4 28 40 46 49 25 13 7) (6 29 15 8 30 41 21 11)*
(10 31 16 34 43 22 37 19) (12 32 42 47 24 38 45 23) (18 35) (20 36 44 48 50 51 26 39)

b) The number of times you must do a perfect shuffle before the deck returns to its original order.

Since there are 6 8-cycles and 1 2-cycle, the shuffle only requires 8 repetions to restore to originial ordering.

2) Top Half of Deck in Right Hand

a) The cyclic notation for these permutations of the deck of 52 cards

The 1-line notation looks as follows [27 1 28 2 29 3 ... 52 26], thus n odd goes to $26 + \frac{n+1}{2}$, and n even goes to $\frac{n}{2}$. Hence the cycle is (1 27 40 20 10 5 29 41 47 50 25 39 46 23 38 19 36 18 9 31 42 21 37 45 49 51 52 26 13 33 43 48 24 12 6 3 28 14 7 30 15 34 17 35 44 22 11 32 16 8 4 2).

b) The number of times you must do a perfect shuffle before the deck returns to its original order.

One would be forced to shuffle 52 times to restore order.

Comment: I had a friend who studied this problem back in high school trying to find a pattern amongst the number of cards in the deck and the number of shuffles needed to restore the order. For the most part this was brute forced on a computer. It wasn't until years later that I realized how challenging this sort of thing is to understand. For the record getting an n-cycle with n cards is pretty rare.

Problem #4

Saturated chains in 1800 intgers partially ordered by divisibility.

The definition offered for saturated chain is rather unclear. It makes referance to maximal and minimal elements. I can only assume these are meant to be understood relative to the partial ordering. Such that an element, a, is minimal iff $z R a \Rightarrow z = a$. Similarly maximal would be that $a R z \Rightarrow z = a$.

Clearly 1 is the minimal element of every chain since 1 divides all integers. However there are numerous possible maximal elements since the only condition is that it no longer divide any number ≤ 1800. This of course is exactly the set of integers = {901, 902, ... 1800}, since any thing ≤ 900 will have 2 times itself in the set of numbers less than 1800.

I don't know how to do this in full generality short of considering the cases associated with all 900 elements, which seems rather unfeasible so I am going to assume she meant to ask how many saturated chains using these numbers contain 1800. Which is a much easier question.

1800 has the prime decomposition $3^2 * 2^3 * 5^2$. In order that we have covering relations, those numebers which relate to 1800 should differ from 1800 in only one factor. Thus in order to find all those numbers which are covered by 1800 we should "turn off" the possible factors of 1800. Thus getting $3 * 2^3 * 5^2 = 600, 3^2 * 2^2 * 5^2 = 900$, $3^2 * 2^3 * 5 = 360$. We tend proceed to remove a single factor of each of these number, etc. This process thus comes down to looking at all the possible permutations of the factors of 1800 such that we are multiplying 2's, 3's and 5's together in distinct orders. Hence the permutation of multisets probelms, showing that there are $\frac{7!}{2!\,3!\,2!}$ ways to make saturated chains.

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  7!
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2! 3! 2!

210
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Just for kicks I'm going to brute force the harder problem of all the maximal elements less than or equal to 1800.

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For[n = 901; t = 0, n ≤ 1800, n++,
 temp = FactorInteger[n]; d = 1; e = 0; For[c = 1, c ≤ Length[temp],
  c++, d = d / temp[[c, 2]]!; e = e + temp[[c, 2]]]; t = t + e! * d];
t

9274
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Problem #5

Rolling 3 Dice to get a total of 5.

a) Distinguishable Dice

You can get a sum of 5 from rolling $<1, 1, 3>$ or $<1, 2, 2>$. Each of which can be rolled 3 differant ways by determining which die the 3 and 1 are on respectively. Thus we have 6 ways to roll 5's. While there are 6^3 ways of rolling 3 independant dice.

Thus the probability of rolling a 5 on 3 dice is $\frac{1}{36}$.

a) Indistinguishable Dice

Since we don't care which die rolls what the above combination represent both possible configurations and we have 2 ways to reach 5. However in order to count the number of total ways to roll indistinguishable dice we have to be careful. Since there is no inherent ordering we might as well use representations sorted from least to greatest. In such a system there are 6 choices for the first number, but we can only choose numbers for the second which are ≥ the first and the third has choices ≥ the second. We can express this as $\sum_{i=1}^{6} \sum_{j=i}^{6} \sum_{k=j}^{6} 1$

$$\sum_{i=1}^{6}\sum_{j=i}^{6}\sum_{k=j}^{6}1$$

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56
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Hence ∃ 56 combinations of possible dice rolls. So we have $\frac{2}{56} = \frac{1}{28}$probability of rolling a 5 on indistinguishable dice.

Problem #6

How many permutations of order two are there on 6 letters?

Permutations of order two must be composed entirely of transpositions, as the second application of the map must undo whatever it did on the first run. ∃ $\binom{6}{2}$ ways to create transpostions of 2 elements. Suppose we have a permutation that is the product of two transpositions. If the transposition are disjoint then they won't interfere with each other and ths is fine. If they are they same then there product is the identity, so we don't want to consider this case. What if they have exactly 1 element in common? Consider $(a\,b)\,(b\,c)$, where $a \neq b \neq c$. We can easily see that this is $(b\,c\,a)$ which is an order 3 permutation and thus not something we wish to include in our description of order 2 permutations.

So the number of single transpositions is $\binom{6}{2}$.

Double transpositions is $\binom{6}{2}\binom{4}{2}\Big/2!$, since the transpositions have no elements in common and it doesn't matter which order they are chosen in.

Triple Transposition are $\binom{6}{2}\binom{4}{2}\binom{2}{2}\Big/3!$.

Hence the total number of order 2 permutations is $\binom{6}{2}+\binom{6}{2}\binom{4}{2}\Big/2!+\binom{6}{4}\binom{4}{2}\binom{2}{2}\Big/3!$

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Binomial[6, 2] * (1 + Binomial[4, 2] * (1 / 2! + Binomial[2, 2] / 3!))
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75
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∴ ∃ 75 order two permutations of 6 letters.

Problem #7

On Permutations

- **a) Find the inverse of [4 5 6 1 3 2] in 1-line notation. Find the inverse of (4 5 6 1 3 2) in cyclic notation. Are they the same?**

1-line inverse = [4 6 5 1 2 3] = (1 4) (2 6 3 5)

Cyclic Inverse = (1 6 5 4 2 3).

Not the Same.

- **b) Find the inverses of [3 5 1 2 4] and [1 4 5 2 3]. Calculate their product and its inverse. What is the relationship between the inverse of the products and the product of the inverses? Prove a general relation.**

$[3\,5\,1\,2\,4]^{-1} = [3\,4\,1\,5\,2]$; $[1\,4\,5\,2\,3]^{-1} = [1\,4\,5\,2\,3]$
$[3\,5\,1\,2\,4][1\,4\,5\,2\,3] = [3\,2\,4\,5\,1]$; $[3\,2\,4\,5\,1]^{-1} = [5\,2\,1\,3\,4]$
$[3\,5\,1\,2\,4]^{-1} * [1\,4\,5\,2\,3]^{-1} = [3\,4\,1\,5\,2][1\,4\,5\,2\,3] = [3\,5\,2\,4\,1]$

It's not really specified but if we look at the multiplication the other direction, ie [1 4 5 2 3][3 5 1 2 4] = [5 3 1 4 2] and the inverse of this is [3 5 2 4 1], which is exactly the same as the product of the inverses above. Hence the hypothesis should be that if σ, τ are permutations then $(\sigma\tau)^{-1} = \tau^{-1}\,\sigma^{-1}$. As proof I offer the following observation:

Given that permutations are associative and identities are unique, we can see that $(\sigma\tau) * (\sigma\tau)^{-1} = e = \sigma\sigma^{-1} = \sigma * e * \sigma^{-1} = \sigma(\tau\tau^{-1})\,\sigma^{-1} = (\sigma\tau)\,(\tau^{-1}\,\sigma^{-1})$. Hence since we have inverses cancelling $\sigma\tau$ gives that $(\sigma\tau)^{-1} = \tau^{-1}\,\sigma^{-1}$. QED

Problem #8

Chapter 5, #3

Obviously the diagonal sums are giving us fibonacci numbers but the question is why shoudl this be so?

Each diagonal sum has the form $\binom{n}{0}$+$\binom{n-1}{1}$+$\binom{n-2}{2}$+...+$\binom{[\frac{n+1}{2}]}{[\frac{n}{2}]}$, where $[x]$ is the greatest integer ≤ x.

WLOG we may treat each sum as $\binom{n}{0}$+$\binom{n-1}{1}$+$\binom{n-2}{2}$+...+$\binom{1}{n-1}$, since all the extra terms we are adding will contibute 0. This is more easily represented as $\sum_{i=0}^{n-1}\binom{n-i}{i}$.

We know by Pascal's formula that $\binom{n}{k}$=$\binom{n-1}{k}$+$\binom{n-1}{k-1}$. Applying this to the sum above we have $\left(\sum_{i=1}^{n-1}\binom{n-1-i}{i}+\binom{n-1-i}{i-1}\right)+\binom{n}{0}$. The extra term comes from our wish to avoid trying the understand what choosing -1 items would mean. Breaking the sum into pieces and reindexing we have $\sum_{j=2}^{n}\binom{n-j}{j-1}+\sum_{i=1}^{n-1}\binom{n-1-i}{i-1}+\binom{n}{0}$. This in turn may be written as $\binom{0}{n-1}+\binom{n-2}{0}+\binom{n}{0}+\sum_{i=2}^{n-1}\binom{n-i}{i-1}+\binom{n-1-i}{i-1}$. Since $\binom{0}{n-1}=0$ and $\binom{n}{0}=1=\binom{n-1}{0}$, we may further rearrange terms to get $\sum_{i=1}^{n-1}\binom{n-i}{i-1}+\binom{n-1-i}{i-1}$. Reindexing gives us $\sum_{i=0}^{n-2}\binom{n-i-1}{i}+\binom{n-i-2}{i}$ which equals $\sum_{i=0}^{(n-1)-1}\binom{(n-1)-i}{i}+\sum_{i=0}^{(n-2)-1}\binom{(n-2)-i}{i}$. Since the only missing term at left is the $\binom{0}{n-2}$ which is of course 0.

Hence the sum over the n^{th} diagonal is equal to the sum over the $(n-1)^{\text{th}}$ diagonal + the sum over the $(n-2)^{\text{th}}$ diagonal. Since we further know that the 0^{th} and 1^{st} diagonals are 1, this is sufficent to conclude that the n^{th} diagonal sum will yield the n^{th} fibonacci number.

The problem asks that we compare this result to the result of Chapter 1, #4, a problem which wasn't assigned. Without elaborate proof I'll say that the numbers are the same. The reason being that a $2\times n$ chessboard can be covered in dominoes exactly as many ways as the sum of the ways to cover $2\times(n-1)$ and $2\times(n-2)$ chessboasds. This is because one can either lay out a dominoe across the top making a $2\times(n-1)$ subboard out of the $2\times n$ board or one can lay down 2 adjacent dominoes oriented along the length of the board making a $2\times(n-2)$ subboard. Any other attempts to reduce the problem will already be counted in one of these cases and these cases are also mutually exclusive, hence their sum gives the number of ways of covering the board. $f(1) = 1,\ f(2) = 2$ thus guaranting that the sums will match up.

Problem #9

Chapter 5, #8

We wish to show that $2^n = \sum_{k=0}^{n} (-1)^k \binom{n}{k} 3^{n-k}$

We know that $(x + y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k\, y^{n-k},\ \forall\, x,\ y$.

So simply choose $x = -1,\ y = 3$ then you have $(-1 + 3)^n = 2^n = \sum_{k=0}^{n} \binom{n}{k} x^k\, y^{n-k} = \sum_{k=0}^{n} \binom{n}{k} (-1)^k\, 3^{n-k}$. Thus we are done.