Math 475

Combinatorics and Graph Theory

HW Set #4

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■ Key to Symbols and Notation

WLOG = Without Loss of Generality

BWOC = By Way of Contradiction

${}_nP_r = P_r^n = P(n, r)$ = Number of Ordered Ways of Choosing r things from a set of n things.

${}_nC_r = C_r^n = C(n, r) = \binom{n}{r} =$ Binomial[n, r] = Choosing r things from a set of n things.

$n!$ = n Factorial

$[x_1\ x_2 \ldots x_n]$ = 1-line notation for the permutation of n elements.

$(x_1\ x_2 \ldots x_m)$ = a permutation cycle of m elements.

$< x_1, x_2, \ldots x_k >$ = multiset of k elements.

$\{x_1, x_2, \ldots x_l\}$ = set of l elements.

$\vee$ = logical OR

$\wedge$ = logical AND

$\ni$ = such that

Problem #1

■ Chapter 4, #41

BWOC, Suppose there exist two distinct partial orders P, Q both having a specified cover relation C on the set X. Since they are distinct orderings $\exists\, a, b \in X$ such that a, b are related by either P or Q and not the other. WLOG suppose $a\,P\,b$ so that $a\,Q\,b$ is false. We shall denote not Q as Θ, and not C as χ. So $a\,\Theta\,b$. $\forall\, x, y \in X$, $x\,C\,y \Rightarrow x\,P\,y \bigwedge x\,Q\,y$. Thus by contrapositive we know that $a\,\chi\,b$ which $\Rightarrow \exists\, c_1 \in X \ni (a\,P\,c_1 \bigwedge c_1\,P\,b)$. By transitivity $(a\,Q\,c_1 \bigwedge c_1\,Q\,b)$ would $\Rightarrow a\,Q\,b$, hence either $a\,\Theta\,c_1 \bigvee c_1\,\Theta\,b \Rightarrow a\,\chi\,c_1 \bigvee c_1\,\chi\,b$. Thus continuing the pattern, given any two elements $\alpha, \beta \in X$ such that $\alpha\,\chi\,\beta \wedge \alpha\,P\,\beta \wedge \alpha\,\Theta\,\beta$ we can find another element $c_i \in X$ such that $\alpha \neq c_i \neq \beta$ and χ, P, Θ are true on either the ordered pair (α, c_i) or (c_i, β) and $\alpha\,P\,c_i\,P\,\beta$. This requires that $\exists$ infinitely many distinct elements between a and b, which is a contradiction since X is finite. Hence P and Q are not distinct and C uniquely determines a partial ordering.

Problem #2

■ Chapter 5, #10

We wish to justify the formula $k\binom{n}{k} = n\binom{n-1}{k-1}$ on combinatorial grounds. If you have a team of k people formed from amongst n possible people then there are $\binom{n}{k}$ to choose the team and k ways to pick a captain amongst them or $k\binom{n}{k}$ ways to pick a team with a captain. If instead you choose the captain first there are n ways to do this and then $\binom{n-1}{k-1}$ ways to pick the remaining team members. Thus $k\binom{n}{k} = n\binom{n-1}{k-1}$. QED.

Problem #3

■ Chapter 5, #20

We want integers such that $m^3 = a\binom{m}{3} + b\binom{m}{2} + c\binom{m}{1}$ using Newton's form of the choose expression we know that this equals $a\,m(m-1)\,\frac{(m-2)}{3!} + b\,\frac{m(m-1)}{2!} + c * m$ $=$ $\frac{a}{6}\,(m^3 - 3\,m^2 + 2\,m) + \frac{b}{2}\,(m^2 - m) + c * m$ $=$ $\frac{a}{6}\,m^3 + \frac{b-a}{2}\,m^2 + \frac{2\,a - 3\,b + 6\,c}{6}\,m$. Thus $a = 6$, $b = 6$, $c = 1$.

Problem #4

Chapter 5, #26

$\sum_{\substack{r,s,t\geq 0 \\ r+s+t=n}} \binom{m_1}{r}\binom{m_2}{s}\binom{m_3}{t} = \binom{m_1+m_2+m_3}{n}$. You have to think of this in the proper way. Consider an unordered basket of n things. You choose the n things from 3 groups containing m_1, m_2, and m_3 items respectively. The number of ways r, s, t may sum to n is exactly the number of different ways in which we may choose the number of items taken from each group. Of course taking items from differant groups are independant processes so they obey the product rule, hence the $\binom{m_1}{r}\binom{m_2}{s}\binom{m_3}{t}$. Taking the sum over all possible r, s, t we thus have consdered all the possible configurations of taking items from each group. This total is thus equivalent to choosing n items from amongst all the groups lumped together, hence the $\binom{m_1+m_2+m_3}{n}$. QED.

Problem #5

Chapter 5, #30

$\{\emptyset \subset \{1\} \subset \{1, 2\} \subset \{1, 2, 3\} \subset \{1, 2, 3, 4\} \subset \{1, 2, 3, 4, 5\}$; $\{2\} \subset \{2, 3\} \subset \{2, 3, 4\} \subset \{2, 3, 4, 5\}$; $\{3\} \subset \{3, 4\} \subset \{3, 4, 5\} \subset \{1, 3, 4, 5\}$; $\{4\} \subset \{4, 5\} \subset \{1, 4, 5\} \subset \{1, 2, 4, 5\}$; $\{5\} \subset \{1, 5\} \subset \{1, 2, 5\} \subset \{1, 2, 3, 5\}$; $\{1, 3\} \subset \{1, 3, 4\}$; $\{2, 4\} \subset \{2, 4, 5\}$; $\{3, 5\} \subset \{1, 3, 5\}$; $\{1, 4\} \subset \{1, 2, 4\}$; $\{2, 5\} \subset \{2, 3, 5\}\}$

Problem #6

Chapter 5, #32

It is important to first realize that the question is actually what is the longest possible antichain of the subsets of $\{1, 2, \ldots, 10\}$ partially ordered by $\subset$. This is because you want the jokes seen on one night not to be a subset of the jokes seen on any other night, and hence not belonging to the same chain. Applying theorem 5.4.3 we know that this $\binom{10}{5}$.

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Binomial[10,5]
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252
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Or in other words it is possible to watch 252 episodes without ever seeing a night's jokes which were a subset of any other night's jokes. :-) That's a lot of comedy. Of course I suppose after hearing all 10, you might think they were repetitive even if he did mix up the order.

Problem #7

Calculations

a) Chapter 5, #36

$$\frac{10!}{3!1!4!0!2!}$$

12600

b) Chapter 5, #37

$$\frac{9!}{3!3!1!2!} 1^3 (-1)^3 2^1 (-2)^2$$

-40320

Problem #8

Chapter 5, #39

The multinomial formula $\binom{n}{n_1\ n_2\ \ldots n_t}$ is exactly the number of permutation of n objects belonging to t classes with repitition numbers $n_1,\ n_2, \ldots,\ n_t$. Suppose instead we count the permutations by first choosing the object in the first position and then choosing the remaining objects for the permutation. We can do this in t different ways since there are t differant types of objects to put in the first position. We will then have $n-1$ positions to fill and one less of the type of object we chose for the first position. This corresponds to the formula. $\binom{n-1}{n_1-1\ n_2\, n_3 \ldots n_t} + \binom{n-1}{n_1\ n_2-1\, n_3 \ldots n_t} + \ldots + \binom{n-1}{n_1\ n_2 \ldots n_t - 1}$. Since we can thus equate the two formulas we are done. QED

Problem #9

Chapter 5, #42

$10^{\frac{1}{3}} = (8+2)^{1/3} = 2*\left(1+\frac{1}{4}\right)^{1/3} = 2*\sum_{k=0}^{\infty}\binom{\frac{1}{3}}{k}\left(\frac{1}{4}\right)^k$

$$2.0*\sum_{k=0}^{\infty}\text{Binomial}[1/3,k]\left(\frac{1}{4}\right)^k$$

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2.15443
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$$10.0^{\frac{1}{3}}$$

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2.15443
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Problem #10

Consider the partially ordered set of {1, 2, ... , 12} ordered by divisibility.

What is the size of the largest antichain?

5, one such antichain is {2, 3, 5, 7, 11}.

Partition the ordered set into a minimal number of chains.

{{1, 3, 6, 12}, {2, 4}, {5, 10}, {7}, {11}}

Partition the set into a minimal number of antichains.

{{2, 3, 5, 7, 11}, {4, 6, 10}, {12}, {1}}