Math 475

Combinatorics and Graph Theory

HW Set #9

By Bobby Rohde
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Key to Symbols and Notation

WLOG = Without Loss of Generality
BWOC = By Way of Contradiction
${}_nP_r = P_r^n = P(n, r) =$ Number of Ordered Ways of Choosing r things from a set of n things.
${}_nC_r = C_r^n = C(n, r) = \binom{n}{r} =$ Binomial[n, r] = Choosing r things from a set of n things.
$n! = n$ Factorial
$[x_1\ x_2\ \ldots\ x_n] =$ 1-line notation for the permutation of n elements.
$(x_1\ x_2\ \ldots\ x_m) =$ a permutation cycle of m elements.
$< x_1, x_2, \ldots x_k > =$ multiset of k elements.
$\{x_1, x_2, \ldots x_l\} =$ set of l elements.
$(x_1, x_2, \ldots, x_n) =$ ordered n-tuple.
$\vee =$ logical OR
$\wedge =$ logical AND
$\ni =$ such that
$a \mid b = a$ divides b

Problem #1

Cycles and Chapter 8, #19

Problem #2

Arranging Numbers in an Ordered Way

Let f be a function from $\{1, 2, 3, \ldots 2n\} \to \{+1, -1\}$, such that $\sum_{i=1}^{2n} f(n) = 0$, ie, there are an equal number of +1's and -1's. Let A and B be sets such that $A \bigcup B = \{1, 2, \ldots 2n\}$ and $f(A) = \{+1\}$, $f(B) = \{-1\}$. Then we can fill a $2 \times n$ board according the rule that the m^{th} item in the top (bottom) row is the m^{th} smallest item in A (B). Since $|A| = |B| = n$ as a result of there construction, we are guaranteed to fill the board. Since it also follows that they disjoint we have a way or arranging $\{1, 2, \ldots, 2n\}$ onto the board with no repeats.

We want the items in the top row to be greater than those in the bottom row, this says that the m^{th} item in A is less than the m^{th} item of B or considering the sequence $< f(i) >_{i=1}^{2n}$, we need the $m^{\text{th}} - 1$ comes after the $m^{\text{th}} + 1$, this means that $\sum_{i=1}^{t} f(i) \geq 0, \ \forall\ t$. We get the ordering of the top and bottom rows for free from the fact that we are ordering A and B with regards to $<$. However sequences of this type are exactly what was discussed in the text and may be generated in C_n ways, where C_n is the n^{th} Catalan number.

Thus we can arrange such a board in C_n ways.

Problem #3

Get out the Crayons

Well if I wanted to be really picky I suppose there are two ways to read this problem depending on what you think of as "different" ways to color the board. One might decide that different means only that the ordered triples formed by the number of squares of each color are distinct, in which case we don't care how the squares are colored. But somehow I don't think this what you really mean.

I'm going to go with the assumption that different considers the patterns of colors as well as the number of each color.

This is a permutation of multisets problem, and pursuant to the discussion of p. 245 in the text the generating function will be $g(x) = \left(x + \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots\right)^2 e^x \ = \left(\frac{e^x - e^{-x}}{2}\right)^2 e^x \ = \frac{1}{4}\left(e^{2x} + e^{-2x} - 2\right) e^x$ $= \frac{1}{4}\left(e^{3x} + e^{-x} - 2e^x\right)$

Thus $h_n = \frac{1}{4}\left(3^n + (-1)^n - 2\right)$.

Problem #4

Chapter 9, #2

See Graphic in Appendix.

Problem #5

Chapter 9, #6

Since one of m and n are even then WLOG we may specify that n is even. BWOC Suppose there exist some $m\times n$ board, with n even and both m and n at least 2, with some pair of forbidden positions on opposite colors such that it can not be covered with dominoes. Consider a board $\mathcal{A}$ which cannot be covered and has m as small as possible such that it cannot be covered.

We can then conclude that the forbidden positions are in rows 1 and m, because otherwise $\mathcal{A}$ would be $(m-1)\times n$ borad plus an $1\times n$ board, the first of which is coverable by assumption and the second is coverable since n is even.

Secondly, the forbidden positions are in the (1 or 2) and (n or $n-1$) columns, because if there was no hole in the first two and last two columns then you could subdivide $\mathcal{A}$ as a $m\times(n-2)$ board and a $m\times n$ board, each of which would be coverable. WLOG we may consider a reflection if neccesary to say that there is a forbidden position in the first two squares of the first row. Since the positions must in fact be on opposite colors you only really have one degree of freedom in the placement of the forbidden position, either the 1st or 2nd square of the first row.

- **Case *m* is even:**

  If m is even then the corners are the same color and placing a forbidden square in the top left position is identical to a 180 degree rotation of placing one in row 1, square 2.

  Thus WLOG we may consider there to be a forbidden square in the top left corner. Tile the board starting in row 1, square 2, and proceeding right along the row until there is only one square left (guaranteed since n is even and one space is forbidden). Place a vertical dominoe. There are now $n-1$ squares open in the second row which is odd. Naturally we tile along that row and conclude with a vertical piece. Continue until row $m-1$. At the end of each odd row we will have placed a tile at the far right vertically down onto the row below. Since $m-1$ is odd we can tile exactly as before an place a vertical tile down onto the m^{th} row, which will give us the lower right corner.

  Since the top left and lower right corners are the same color we know that the second forbidden position is in the row m, square $n-1$ position. Now we have filled all available squares except for the first $n-2$ squares of the m^{th} row. However $1\times(n-2)$ boards are clearly coverable since n is even.

  Thus we have covered this board, contradicting our assumption that it is uncoverable and showing that there are no uncoverable boards of this type with m, n both even.

- **Case *m* is odd, row 1, square 1 forbidden:**

  As before the essential strategy is to walk along each row, starting with the first until you can no longer do so, and then place a domino vertically going down a level. Since each row is of even length, and the first position is forbidden, then you will place a vertical piece in the last square. Similarly there is a vertical piece going down at the end of each odd row. Since in this configuration we will get the other forbidden position in the row m, square n, position, and m is odd, this means that ordinarily we would reach the end of row m and have one space left, but since that last space is forbidden, it's all good and we have covered the board.

  Thus this case is coverable, and only one case remains.

- **Case *m* is odd, row 1, square 2 forbidden:**

  Flip across the a vertical line down the middle so that position row 1, square $n-1$ is forbidden. The first row, squares $n-2$ can then be filled with horizontal dominoes and similarly the last row, last $n-2$ squares. The remainder of the board can then be filled identically to the case above.

  Hence we have covered all possible uncoverable cases. Thus no such board can be uncoverable according to the restrictions given above. Thus all boards with two pieces removed pursuant to the previous restrictions, will be coverable.

Problem #6

Chapter 9, #9

"There is a one-to-one correspondence between sets of non-attacking rooks on the board and matchings of the associated bipartite graph" - p. 298

Apply Theorem 9.2.5.

It works.

Problem #7

Chapter 9, #11

All 7 positions can be filled. One possible matching is $\{\{x_9, y_1\}, \{x_7, y_2\}, \{x_8, y_3\}, \{x_3, y_4\}, \{x_{10}, y_5\}, \{x_1, y_6\}, \{x_5, y_7\}\}$.

Problem #8

Chapter 9, #14

If $x \notin A_i, \ \forall \ i \neq 1$, then given any SDR, we may choose to substitute x to represent A_1, since we are guaranteed that no other set will be using it as a representative.

Suppose $x \in A_i$, for some i, then given some SDR then if x does not represent any other set then we may substitute x as the representative for A_1. However if it represents some other set then we already have x in the SDR so we are done.

Thus there always exists an SDR with x in it.

Let $\mathcal{A} = (\{x, a\}, \{x\}, \{b\})$, then there exist an SDR with (a, x, b), but x can not be used for the first set since it must be used for the second.