Real Analysis - Math 630

Homework Set #1

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9-7-00

Problem 1

■ **If A and B are two sets in $\mathfrak{M}$ with $A \subset B$, then $mA \leq mB$.**

■ **Proof**

Since $\mathfrak{M}$ is a σ-algebra, we know that $C = B \bigcap \tilde{A} \in \mathfrak{M}$, with $C \bigcap A = \emptyset$ and $C \bigcup A = B$. Hence $mB = m(C \bigcup A) = mC + mA \geq mA$. $\therefore mB \geq mA$, QED.

Problem 2

■ **Let $<E_n>$ be any sequence of sets in $\mathfrak{M}$. Then $m(\bigcup E_n) \leq \sum mE_n$.**

■ **Proof**

By Proposition 1.2, $\exists$ a sequence $<A_n>$ of sets in $\mathfrak{M}$ with $\bigcup E_n = \bigcup A_n$ and $A_n \bigcap A_m = \emptyset$, $\forall$ n $\neq$ m. Thus $m(\bigcup E_n) = m(\bigcup A_n) = \sum mA_n$. If we construct A_n from E_n according to the algorithim used in the proof of Proposition 1.2 then we have $A_n \subset E_n$ $\forall$ n, and from Problem 1 therefore $mA_n \leq mE_n$. Hence we have $\sum mA_n \leq \sum mE_n \Rightarrow m(\bigcup E_n) \leq \sum mE_n$. QED.

Problem 3

If there is a set A in $\mathfrak{M}$ such that $mA < \infty$, then $m\emptyset = 0$.

Proof

By Way of Contradiction (BWOC). Suppose $m\emptyset = \alpha \neq 0$, and let $mA = \beta < \infty$. $A \cap \emptyset = \emptyset$, hence A and $\emptyset$ are disjoint. Thus by Property 3, $m(A \cup \emptyset) = mA + m\emptyset = \beta + \alpha$. But $A \cup \emptyset = A$, so $m(A \cup \emptyset) = mA = \beta$. Thus $\beta = \beta + \alpha > \beta$. Which is a contradiction, hence $m\emptyset = 0$. QED.

Problem 4

Let nE be ∞ for an infinite set E and equal to the number of elements in E for a finite set. Show that n is a countably additive set function that is translation invariant and defined for all sets of real numbers.

Proof

Countably Additive

We need to show that $\forall$ disjoint sequences $<A_i>$ of sets in $\mathbb{R}$, $n(\bigcup A_i) = \sum nA_i$.

If A_i is infinite for any i then $n(\bigcup A_i) = \sum nA_i = \infty$, so we may as well assume that A_i is finite $\forall\, i$. However since A_i is disjoint that implies that the number of elements in $A_i \cup A_j = nA_i + nA_j$, $\forall$ i $\neq$ j. Thus by repeated applications it is clear that $n(\bigcup A_i)$ must equal $\sum nA_i$. QED.

Translation Invariant

We need to show that $n(A + \mathrm{y}) = nA$ $\forall$ y.

Clearly the definition of $A + \mathrm{y} = \{\mathrm{x} + \mathrm{y} : \mathrm{x} \in A\}$, establishes a 1-1 correspondance between elements of A and elements of A + y, thus A and A + y must have the same number of elements and hence $n(A + \mathrm{y}) = nA$ $\forall$ y. QED

Defined on All Sets of $\mathbb{R}$

Clearly it is in the nature of sets that they must have either an infinite number of elements or exactly one well-defined finite number of elements so it is impossible to construct a set on $\mathbb{R}$ which does not have a unique value under operation by n. Thus all sets on $\mathbb{R}$ are measurable by n. QED.

Problem 5

- **Let A be the set of rational numbers between 0 and 1, and let $\{I_n\}$ be a finite collection of open intervals covering A. Show that $\sum l(I_n) \geq 1$.**

- **Proof**

Let (a_n, b_n) denote the interval I_n with end points $a_n < b_n$.

BWOC Suppose $0 < a_n$, $\forall\ n$, then since rationals are dense, $\exists$ a rational number γ between 0 and $\min\{a_n\}$, not covered by I_n, which is a contradiction. So we know $\exists\ n$ such that $a_n \leq 0$. Similarly we must have an m such that $b_m \geq 1$.

BWOC Suppose $\exists\ I_n$ with $b_n < 1$ such that $\forall$ m ≠ n, $a_m \leq b_n < b_m$ is false $\Rightarrow \exists$ an open interval $C = (b_n, \min(\{a_n\text{: } a_n > b_n\} \cup \{1\}))$, which is not in $\bigcup I_n$. But since the rational numbers are dense, $\exists$ a rational number in C, which contradicts the fact that $\{I_n\}$ covers A. Thus $\forall\ I_n$ with $b_n < 1$, $\exists\ m$ such that $a_m \leq b_n < b_m$.

Consider $T = (\bigcup I_n) \cup \{b_n\} \cup \{a_n\}$. Clearly T covers A since $\bigcup I_n$ covers A. Furthermore since $a_m \leq b_n < b_m$ is true under the conditions just stated, this implies that including $\{a_n\}$ and $\{b_n\}$ will close any gaps between intervals. Since there are a finite number of open intervals, we are adding only a finite number of points to T, and hence $m^*T = \sum\ l(I_n)$. But T is now a continuous interval with $(0, 1) \subseteq T$, using the fact that $\exists\ n, m$ such that $a_n \leq 0$ and $b_m \geq 1$. Thus $m^*T \geq 1 - 0 = 1$. $\therefore \sum l(I_n) \geq 1$. QED.

Problem 6

- **Prove that: Given any set A and any $\epsilon > 0$, $\exists$ an open set O such that $A \subset O$ and $m^*O \leq m^*A + \epsilon$. There is a $G \in G_\delta$ such that $A \subset G$ and $m^*A = m^*G$.**

- **Proof**

Since $\inf\limits_{A \subset \bigcup I_n} \sum\ l(I_n) = m^*A$ we know by definition of infimum that $\exists$ a collection of open intervals $\{I_n\}$ where $\sum\ l(I_n) \leq m^*A + \epsilon\ \forall\ \epsilon > 0$. Thus we have that $O = \bigcup I_n$ is an open set with $m^*O = \sum\ l(I_n) \leq m^*A + \epsilon$. Which gives us the first part of the Proposition

For the second part we merely have to consider $G = \bigcap_{n=1}^{\infty} O_{\frac{1}{n}}$, where O_ϵ is the O associated with a specified ϵ in the first part. Since $A \subset O_\epsilon\ \forall\ \epsilon > 0$, we know that $A \subseteq G$. Also since G is an intersection of a countable collection of open intervals, $G \in G_\delta$. Finally since $m^*A \le m^*O \le m^*A + \epsilon$, with $\epsilon \to 0$ in our construction of G, we can therefore conclude that $m^*A = m^*G$. QED.

Problem 7

■ Prove that m^* is translation invariant.

■ Proof

Let A be a set in $\mathbb{R}$, then we need to show that $m^*(A + \mathrm{y}) = m^*A \ \forall\ \mathrm{y}$.

Let $\{I_n\}$ be a countable collection of open intervals that cover A, then $m^*A = \inf\limits_{A \subset \bigcup I_n} \sum l(I_n)$. Clearly $\{I_n + \mathrm{y}\}$ will cover $A + \mathrm{y}$, with $m^*(A + \mathrm{y}) = \inf\limits_{A \subset \bigcup I_n} \sum l((I_n + y))$, but $l((A_n + y)) = l(A_n)$, so $\inf\limits_{A \subset \bigcup I_n} \sum l(I_n) = \inf\limits_{A \subset \bigcup I_n} \sum l((I_n + y)) \Rightarrow m^*(A + \mathrm{y}) = m^*A$. QED.

Problem 8

■ Prove that if $m^*A = 0$, then $m^*(A \bigcup B) = m^*B$.

■ Proof

If $A \subset B$ then $A \bigcup B = B$ which is trivially true. So we may assume $A \not\subset B$. Let $\{I_n\}$ denote open covers of A, and $\{J_n\}$ denote open covers of B. Thus $\{I_n\} \bigcup \{J_n\}$ will cover $A \bigcup B$. Furthermore $m^*(A \bigcup B) \le \inf\limits_{A \bigcup B \subset \bigcup (I_n \bigcup J_n)} \sum (l(I_n) + l(J_n)) = \inf\limits_{A \bigcup B \subset \bigcup (I_n \bigcup J_n)} (\sum l(I_n)) + (\sum l(J_n)) = \inf\limits_{A \subset \bigcup I_n} \sum l(I_n) + \inf\limits_{B \subset \bigcup J_n} \sum l(J_n) = m^*A + m^*B = m^*B$.

But also $B \subset A \bigcup B$ so $m^*B \le m^*(A \bigcup B)$. Thus we must have that $m^*(A \bigcup B) = m^*B$. QED.