Problem 1

If $A$ and $B$ are two sets in $\mathcal{M}$ with $A \subset B$, then $mA \leq mB$.

Proof
Since $\mathcal{M}$ is a $\sigma$-algebra, we know that $C = B \cap \bar{A} \in \mathcal{M}$, with $C \cap A = \emptyset$ and $C \cup A = B$. Hence $mB = m(C \cup A) = mC + mA \geq mA$. $\therefore mB \geq mA$, QED.

Problem 2

Let $\langle E_n \rangle$ be any sequence of sets in $\mathcal{M}$. Then $m(\bigcup E_n) \leq \sum mE_n$.

Proof
By Proposition 1.2, $\exists$ a sequence $\langle A_n \rangle$ of sets in $\mathcal{M}$ with $\bigcup E_n = \bigcup A_n$ and $A_n \cap A_m = \emptyset$, $\forall n \neq m$. Thus $m(\bigcup E_n) = m(\bigcup A_n) = \sum mA_n$. If we construct $A_n$ from $E_n$ according to the algorithm used in the proof of Proposition 1.2 then we have $A_n \subset E_n$ $\forall n$, and from Problem 1 therefore $mA_n \leq mE_n$. Hence we have $\sum mA_n = \sum mE_n \Rightarrow m(\bigcup E_n) \leq \sum mE_n$. QED.
Problem 3

- If there is a set $A$ in $\mathcal{M}$ such that $mA < \infty$, then $m\emptyset = 0$.

**Proof**

By Way of Contradiction (BWOC). Suppose $m\emptyset = \alpha \neq 0$, and let $mA = \beta < \infty$. $A \cap \emptyset = \emptyset$, hence $A$ and $\emptyset$ are disjoint. Thus by Property 3, $m(A \cup \emptyset) = mA + m\emptyset = \beta + \alpha$. But $A \cup \emptyset = A$, so $m(A \cup \emptyset) = mA = \beta$. Thus $\beta = \beta + \alpha > \beta$. Which is a contradiction, hence $m\emptyset = 0$. QED.

Problem 4

- Let $nE$ be $\infty$ for an infinite set $E$ and equal to the number of elements in $E$ for a finite set. Show that $n$ is a countably additive set function that is translation invariant and defined for all sets of real numbers.

**Proof**

- **Countably Additive**
  
  We need to show that $\forall$ disjoint sequences $<A_i>$ of sets in $\mathbb{R}$, $n(\bigcup A_i) = \sum nA_i$.

  If $A_i$ is infinite for any $i$ then $n(\bigcup A_i) = \sum nA_i = \infty$, so we may as well assume that $A_i$ is finite $\forall$ $i$. However since $A_i$ is disjoint that implies that the number of elements in $A_i \cup A_j = nA_i + nA_j$, $\forall$ $i \neq j$. Thus by repeated applications it is clear that $n(\bigcup A_i)$ must equal $\sum nA_i$. QED.

- **Translation Invariant**

  We need to show that $n(A + y) = nA $ $\forall$ $y$.

  Clearly the definition of $A + y = \{x + y : x \in A\}$, establishes a 1-1 correspondence between elements of $A$ and elements of $A + y$, thus $A$ and $A + y$ must have the same number of elements and hence $n(A + y) = nA $ $\forall$ $y$. QED

- **Defined on All Sets of $\mathbb{R}$**

  Clearly it is in the nature of sets that they must have either an infinite number of elements or exactly one well-defined finite number of elements so it is impossible to construct a set on $\mathbb{R}$ which does not have a unique value under operation by $n$. Thus all sets on $\mathbb{R}$ are measurable by $n$. QED.
Problem 5

- Let $A$ be the set of rational numbers between 0 and 1, and let $\{I_n\}$ be a finite collection of open intervals covering $A$. Show that $\sum l(I_n) \geq 1$.

- Proof

  Let $(a_n, b_n)$ denote the interval $I_n$ with end points $a_n < b_n$.

  BWOC Suppose $0 < a_n, \forall n$, then since rationals are dense, $\exists$ a rational number $\gamma$ between 0 and $\min\{a_n\}$, not covered by $I_n$, which is a contradiction. So we know $\exists n$ such that $a_n \leq 0$. Similarly we must have an $m$ such that $b_m \geq 1$.

  BWOC Suppose $\exists I_n$ with $b_n < 1$ such that $\forall m \neq n, \ a_m \leq b_n < b_m$ is false $\Rightarrow \exists$ an open interval $C = (b_n, \min\{\{a_n: a_n > b_n\} \cup \{1\}\})$, which is not in $\bigcup I_n$. But since the rational numbers are dense, $\exists$ a rational number in $C$, which contradicts the fact that $\{I_n\}$ covers $A$. Thus $\forall I_n$ with $b_n < 1$, $\exists m$ such that $a_m \leq b_n < b_m$.

  Consider $T = (\bigcup I_n) \cup \{b_n\} \cup \{a_n\}$. Clearly $T$ covers $A$ since $\bigcup I_n$ covers $A$. Furthermore since $a_m \leq b_n < b_m$ is true under the conditions just stated, this implies that including $\{a_n\}$ and $\{b_n\}$ will close any gaps between intervals. Since there are a finite number of open intervals, we are adding only a finite number of points to $T$, and hence $m^*T = \sum l(I_n)$. But $T$ is now a continuous interval with $(0, 1) \subseteq T$, using the fact that $\exists n, m$ such that $a_n \leq 0$ and $b_m \geq 1$. Thus $m^*T \geq 1 - 0 = 1$. $\therefore \sum l(I_n) \geq 1$. QED.

Problem 6

- Prove that: Given any set $A$ and any $\epsilon > 0$, $\exists$ an open set $O$ such that $A \subseteq O$ and $m^*O \leq m^*A + \epsilon$. There is a $G \in G_\delta$ such that $A \subseteq G$ and $m^*A = m^*G$.

- Proof

  Since $\inf_{A \subseteq \bigcup I_n} \sum l(I_n) = m^*A$ we know by definition of infimum that $\exists$ a collection of open intervals $\{I_n\}$ where $\sum l(I_n) \leq m^*A + \epsilon \forall \epsilon > 0$. Thus we have that $O = \bigcup I_n$ is an open set with $m^*O = \sum l(I_n) \leq m^*A + \epsilon$. Which gives us the first part of the Proposition

  For the second part we merely have to consider $G = \bigcap_{n=1}^\infty O_{\epsilon_n}$, where $O_{\epsilon}$ is the $O$ associated with a specified $\epsilon$ in the first part. Since $A \subseteq O_{\epsilon} \forall \epsilon > 0$, we know that $A \subseteq G$. Also since $G$ is an
intersection of a countable collection of open intervals, \( G \subseteq G \). Finally since \( m^* A + m^* O \leq m^* A + \epsilon \), with \( \epsilon \to 0 \) in our construction of \( G \), we can therefore conclude that \( m^* A = m^* G \). QED.

**Problem 7**

**Prove that \( m^* \) is translation invariant.**

**Proof**

Let \( A \) be a set in \( \mathbb{R} \), then we need to show that \( m^*(A + y) = m^* A \) \( \forall y \).

Let \( \{ I_n \} \) be a countable collection of open intervals that cover \( A \), then \( m^* A = \inf \sum_{A \subset \bigcup I_n} l(I_n) \).

Clearly \( \{ I_n + y \} \) will cover \( A + y \), with \( m^*(A + y) = \inf \sum_{A \subset \bigcup I_n} l(I_n + y) \), but \( l((A_n + y)) = l(A_n) \), so \( \inf \sum_{A \subset \bigcup I_n} l(I_n) = \inf \sum_{A \subset \bigcup I_n} l(I_n + y) \Rightarrow m^*(A + y) = m^* A \) \( \Rightarrow \inf \sum_{A \subset \bigcup I_n} l(I_n) = m^* A \). QED.

**Problem 8**

**Prove that if \( m^* A = 0 \), then \( m^*(A \cup B) = m^* B \).**

**Proof**

If \( A \subset B \) then \( A \cup B = B \) which is trivially true. So we may assume \( A \nsubseteq B \). Let \( \{ I_n \} \) denote open covers of \( A \), and \( \{ J_n \} \) denote open covers of \( B \). Thus \( \{ I_n \} \cup \{ J_n \} \) will cover \( A \cup B \). Furthermore

\[
m^*(A \cup B) \leq \inf_{A \cup B \subset \bigcup \{ I_n \cup J_n \}} \sum (l(I_n) + l(J_n)) = \inf_{A \cup B \subset \bigcup \{ I_n \cup J_n \}} (\sum l(I_n)) + (\sum l(J_n)) = \inf_{A \subset \bigcup I_n} \sum l(I_n) + \inf_{B \subset \bigcup J_n} \sum l(J_n) = m^* A + m^* B = m^* B.
\]

But also \( B \subset A \cup B \) so \( m^* B \leq m^*(A \cup B) \). Thus we must have that \( m^*(A \cup B) = m^* B \). QED.