Real Analysis - Math 630

Homework Set #10 - Chapter 6

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11-17-00

Problem 7

- **a) For $1 \le p < \infty$, we denote l^p the space of all sequences $< \xi_\nu >_{\nu=1}^{\infty}$ such that $\sum_{\nu=1}^{\infty} (|\xi_\nu|)^p < \infty$. Prove the Minkowski ineqality for sequences**

  $\| < \xi_\nu + \eta_\nu > \|_p \le \| < \xi_\nu > \|_p + \| < \eta_\nu > \|_p$.

  Here we have $1 \le p \le \infty$,

  $(\| < \xi_\nu > \|_p)^p = \sum_{\nu=1}^{\infty} (|\xi_\nu|)^p$ and $\| < \eta_\nu > \|_\infty = \sup |\eta_p|$

- **Proof**

  - **Case $1 \le p < \infty$**

$\| < \xi_\nu > \|_p = (\sum_{\nu=1}^{\infty} (|\xi_\nu|)^p)^{\frac{1}{p}} \Rightarrow \| < \xi_\nu + \eta_\nu > \|_p = (\sum_{\nu=1}^{\infty} (|\xi_\nu + \eta_\nu|)^p))^{\frac{1}{p}}$.

If either ξ_ν or η_ν has norm 0, the proof is trivial so we may assume that each has norm greater than 1 and then define real numbers $\alpha, \beta > 0$ such that $\| < \frac{\xi_\nu}{\alpha} > \|_p = 1$, $\| < \frac{\eta_\nu}{\beta} > \|_p = 1$.

$(\sum_{\nu=1}^{\infty} (|\xi_\nu + \eta_\nu|)^p))^{\frac{1}{p}} \le (\sum_{\nu=1}^{\infty} (|\xi_\nu| + |\eta_\nu|)^p))^{\frac{1}{p}} = \left(\sum_{\nu=1}^{\infty} \left(\alpha \left| \frac{\xi_\nu}{\alpha} \right| + \beta \left| \frac{\eta_\nu}{\beta} \right|\right)^p\right)\right)^{\frac{1}{p}} = \left(\sum_{\nu=1}^{\infty} (\alpha+\beta)^p \left(\left(\frac{\alpha}{\alpha+\beta}\right) \left| \frac{\xi_\nu}{\alpha} \right| + \left(\frac{\beta}{\alpha+\beta}\right) \left| \frac{\eta_\nu}{\beta} \right|\right)^p\right)\right)^{\frac{1}{p}} \le \left(\sum_{\nu=1}^{\infty} (\alpha+\beta)^p \left(\left(\frac{\alpha}{\alpha+\beta}\right) \left(\left| \frac{\xi_\nu}{\alpha} \right|\right)^p + \left(\frac{\beta}{\alpha+\beta}\right) \left(\left| \frac{\eta_\nu}{\beta} \right|\right)^p\right)\right)^{\frac{1}{p}}$, since $p \ge 1$ and the quantities inside the distribution are each less than 1.

However expnading this expression and summing we see that it is in turn $= (\alpha + \beta) * \rho$, with $\rho \le 1$. So $(\sum_{\nu=1}^{\infty} (|\xi_\nu + \eta_\nu|)^p))^{\frac{1}{p}} \le \alpha + \beta = \| < \xi_\nu > \|_p + \| < \eta_\nu > \|_p$. QED.

- **Case $p = \infty$**

  $\| <\eta_v + \xi_v> \|_\infty = \sup|\xi_v + \eta_p| \leq \sup (|\xi_v| + |\eta_v|) \leq \sup |\xi_v| + \sup |\eta_v| = \| <\xi_v> \|_\infty + \| <\eta_v> \|_\infty$. QED.

b) Show that if $<\xi_v> \in l^p$ and $<\eta_v> \in l^q$ with $\frac{1}{p} + \frac{1}{q} = 1$, then $\sum_{v=1}^\infty |\xi_v * \eta_v| \leq \| <\xi_v> \|_p * \| <\eta_v> \|_q$

Proof

- **Case $p = 1$, $q = \infty$**

  $\sum_{v=1}^\infty |\xi_v * \eta_v| \leq \sum_{v=1}^\infty |\xi_v| * \sup|\eta_v| = \sum_{v=1}^\infty |\xi_v| * \| <\eta_v> \|_\infty = \| <\xi_v> \|_1 * \| <\eta_v> \|_\infty$.

- **Case $1 < p < \infty$**

  Define $\alpha_v = \eta_v^{\frac{q}{p}}$. We may chosoe to assume that ξ_v and η_v are positive since the norm is dependant only on their absolute value. By Lemma 3 $\Rightarrow p * t * \xi_v * \eta_v = p * t * \xi_v * \alpha_v^{p-1} \leq (\alpha_v + t * \xi_v)^p - \alpha_v^p$

  Summing both sides. $\sum_{v=1}^\infty p * t * \xi_v * \eta_v \leq \sum_{v=1}^\infty (\alpha_v + t * \xi_v)^p - \alpha_v^p = \| <\alpha_v + t * \xi_v> \|_p^p - \| <\alpha_v> \|_p^p \leq (\| <\alpha_v> \|_p + \| <t * \xi_v> \|_p)^p - \| <\alpha_v> \|_p^p$

  Differentiating with respect to t we have that $\sum_{v=1}^\infty p * \xi_v * \eta_v \leq p * \| <\xi_v> \|_p * \| <\alpha_v> \|_p^{p-1} = \| <\xi_v> \|_p * \| <\eta_v> \|_q$. QED.

Problem 10

Let $<f_n>$ be a sequence of functions in L^∞. Prove that $<f_n>$ converges to f in L^∞ if and only if there is a set E of measure 0 such that f_n converges uniformly to f on $\tilde{E}$.

Proof

$f_n \to f$ in L^∞ imples $\| f_n - f \|_\infty \to 0$. Thus $\forall\ \epsilon \to 0\ \exists\ N$ such that $\forall\ n > N$, $\| f_n - f \|_\infty < \epsilon$. $\Rightarrow$ except on a set E of measure 0, $|f_n - f| \leq \epsilon$. Thus f_n is uniformly converges to f on $\tilde{E}$.

If f_n uniformly converges to f except on a set E of measure 0 then $\forall\ \epsilon > 0$, $\exists\ N$ such that $\forall\ x \in \tilde{E}$ and $n > N$, $|f_n(x) - f(x)| < \epsilon$. $\Rightarrow$ except on E, $|f(x)| < |f_n(x)| + \epsilon \leq \text{ess sup } f_n(x) + \epsilon$. Thus f is bounded a.e. and hence f is in L^∞.

Problem 11

■ Prove that L^∞ is complete.

■ Proof

In order to show this we need that every Cauchy sequence of functions $< f_n >$ in L^∞ converges to some f in L^∞.

Given $\epsilon > 0$, $\exists$ N such that $\forall\ m, n > N$, $\| f_n - f_m \| < \epsilon$, by definition of Cauchy. Since this is L^∞ norm we know that $f_n - f_m$ is bounded a.e. by ϵ.

Define the pointwise limit $f(x) = \begin{cases} \lim_{n\to\infty} f_n(x), & \text{if the limit exists} \\ \infty, & \text{otherwise} \end{cases}$.

It suffices to show that ess sup $f < \infty$, since if f_n converges then it must converge to f a.e.

Since $f_n - f_m$ is bounded a.e. by ϵ, except on a set M of measure 0. However if we consider the collection of all such $M\ \forall\ m > n$, we have a countable collection of sets of measure zero so there union is also a set of measure 0. Thus $|f_n(x) - f(x)| < \epsilon$ if $\lim_{n\to\infty} f_n(x)$ exists. Since f_n is in L^∞ it is bounded by some ess sup α. Then where the limit exists, f is bounded by $\alpha + \epsilon$.

Thus we only need to show that the limit must exist upto a set of measure 0.

Problem 16

- **Let $< f_n >$ be a sequence of functions in L^p, $1 \le p < \infty$, which converge a.e. to a function f in L^p. Show that $< f_n >$ converges to f in L^p if and only if $\| f_n \| \to \| f \|$.**

- **Proof**

If $\| f_n \| \to \| f \| \Rightarrow \| f_n \| - \| f \| \to 0 \Rightarrow \left| \left(\int (|f_n|)^p\right)^{\frac{1}{p}} - \left(\int (|f|)^p\right)^{\frac{1}{p}} \right| \ge \left| \left(\int (|f_n|)^p - (|f|)^p\right)^{\frac{1}{p}} \right| \to 0$, since $p \ge 1$. However $|| f_n|^p - (|f|)^p | \ge |(|f_n| - |f|)^p|$, for $p \ge 1$

$< f_n >$ converges to f in $L^p \Rightarrow \| f_n - f \| \to 0 \Leftrightarrow \left(\int (|f_n - f|)^p\right)^{\frac{1}{p}} \to 0$. However, $(|f_n - f|)^p \ge (|f_n| - |f|)^p$.

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Demonstrate Proposition 8 fails at p = ∞.

- **Show that $\exists\, f, g \in L^\infty$, such that $\forall$ step functions ϕ and continuous functions ψ, $\| f - \phi \|_\infty \ge \epsilon > 0$ and $\| g - \psi \|_\infty \ge \epsilon > 0$.**

- **Counter-example on step functions.**

Define $f(x) = \begin{cases} \sin(\frac{1}{x}), & x \ne 0 \\ 0, & x = 0 \end{cases}$.

Clearly $f(x)$ is bounded by 1 and thus in L^∞. Consider a step function ϕ on partition $\Delta = \{0 = \xi_0 < \xi_1 < \xi_2 < \ldots < \xi_N = 1\}$. Let δ denote the size of the smallest interval in the partition. Consider intervals of the form $A_n = \left[\frac{1}{n*\pi}, \frac{1}{(n+1)*\pi}\right]$, for $n \in \mathbb{N}$. Clearly $A_n \subset [0, 1]$, and $\exists$ infinitely many A_n of length $< \delta$. So choose n, m such that $A_n \subset [\xi_m, \xi_{m+1}]$ and length $A_n < \delta$. Clearly by construction $|f(A_n)| = [0, 1]$, so regardless of the value of ϕ over $[\xi_m, \xi_{m+1}]$, $|f - \phi| \frac{\ge 1}{2}$ over A_n. Hence $\| f - \phi \|_\infty \ge \frac{1}{2}$. Thus showing the counter example to Proposition 8.

■ Counter-example on continuous functions.

Define $g(x) = \begin{cases} 0, & x \leq \frac{1}{2} \\ 1, & x > \frac{1}{2} \end{cases}$.

Clearly g is in L^∞. Consider a continuous function ψ approximating g. By continuity at $x = \frac{1}{2}$, $\forall\, \epsilon > 0\ \exists\, \delta > 0$ such that $\forall\ |x - \frac{1}{2}| < \delta$, $|\psi(x) - \psi(\frac{1}{2})| < \epsilon$.

Suppose that $\psi(\frac{1}{2}) = 0$, then choose $\epsilon = \frac{1}{2}$ and $\exists$ the interval $(\frac{1}{2}, \frac{1}{2} + \delta)$ where $\psi(x) < \frac{1}{2}$, yet $g = 1$ over this interval so $\| g - \psi \|_\infty > \frac{1}{2}$.

Suppose that $\psi(\frac{1}{2}) \neq 0$, then choose $\epsilon = \frac{\psi(\frac{1}{2})}{2}$ and $\exists$ the interval $(\frac{1}{2} - \delta, \frac{1}{2})$ where $\psi(x) > \frac{\psi(\frac{1}{2})}{2}$. So $\| g - \psi \|_\infty \geq \frac{\psi(\frac{1}{2})}{2}$. Thus $\| g - \psi \|_\infty > 0$. Thus showing the counter example for Proposition 8 at $p = \infty$. QED.