Real Analysis - Math 630

Homework Set #11 - Chapter 6 & 11

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Problem 6.21

- **a) Let g be an integrable function on [0, 1]. Show that $\exists$ a bounded measurable function f such that $\| f \| \neq 0$ and**

  $\int f * g = \| g \|_1 * \| f \|_\infty$

- **Proof**

  - **Case $\| g \|_1 \neq 0$**

    Let $f(x) = \text{sgn}(g(x))$, where $\text{sgn}(\alpha) = \begin{cases} \frac{|\alpha|}{\alpha}, & \alpha \neq 0 \\ 0, & \alpha = 0 \end{cases}$. Since $\| g \|_1 \neq 0$, that implies that $m\{x : g(x) \neq 0\} > 0 \Rightarrow \text{ess sup f(x)} = 1$.

    Hence $\int f * g = \int g * \text{sgn}(g) = \int |g| = \| g \|_1 = \| g \|_1 * \| f \|_\infty$. QED

  - **Case $\| g \|_1 = 0$**

    Let $f(x) = \alpha > 0$, $\forall\ x$. Then $\| f \|_\infty = \alpha$. $\int f * g = \alpha * \int g \leq \alpha * \int |g| = 0\ \forall$ finite α. Thus $\int f * g = \| g \|_1 * \| f \|_\infty = 0$. QED.

- **b) Let g be a bounded measurable function. Show that $\forall\ \epsilon > 0\ \exists$ an integrable function f such that**

$$\int f * g \geq (\| g \|_{\infty} - \epsilon) * \| f \|_1$$

- **Proof**

By definition ess sup g(x) = $\inf\{M : m\{t : f(t) > M\} = 0\}$. Given $\epsilon > 0$. Let $E = \{x : g(x) > \| g \|_{\infty} - \epsilon\}$. Thus $mE > 0$. Let $f(x) = \chi_E$. Then $\int f * g = \int \chi_E * g \geq \int \chi_E * (\| g \|_{\infty} - \epsilon)$ by definition of E. Hence we have this $= mE * (\| g \|_{\infty} - \epsilon) = (\| g \|_{\infty} - \epsilon) * \| f \|_1$. Thus $\int f * g \geq (\| g \|_{\infty} - \epsilon) * \| f \|_1$. QED.

Problem 6.22

- **Find a representation for the bounded linear functionals on l^p, $1 \leq p < \infty$.**
- **Construction**

Problem 6.24

- **Show that the element g in L^p given by Theorem 13 is unique.**
- **Proof**

Problem 11.7

- **Prove Proposition 4: If $(X, \mathcal{B}, \mu)$ is a measure space, then we can find a complete measure space $(X, \mathcal{B}_0, \mu_0)$ such that**

  i. $\mathcal{B} \subset \mathcal{B}_0$.
  ii. $E \in \mathcal{B} \Rightarrow \mu E = \mu_0 E$
  iii. $E \in \mathcal{B}_0 \Leftrightarrow E = A \bigcup B$ where $B \in \mathcal{B}$ and $A \subset C,\ C \in \mathcal{B}, \mu C = 0$.

- **Proof**

First we must show that the $\mathcal{B}_0$ defined by (iii) is in fact a σ-algebra. Thus we need that $\forall$ $E_1, E_2 \in \mathcal{B}_0, \Rightarrow E_1 \bigcup E_2 \in \mathcal{B}_0$ and $\tilde{E}_1 \in \mathcal{B}_0$ and the union of any countable collection of sets in $\mathcal{B}_0$ is in $\mathcal{B}_0$.

Clearly $E_1 \in \mathcal{B}_0 \Rightarrow E_1 = A_1 \bigcup B_1$ where $B_1 \in \mathcal{B}$ and $A_1 \subset C_1,\ C_1 \in \mathcal{B},\ \mu C_1 = 0$. And similarly for E_2. Thus $E_1 \bigcup E_2 = (A_1 \bigcup A_2) \bigcup (B_1 \bigcup B_2)$. And we know that $(B_1 \bigcup B_2) \in \mathcal{B}$, since $\mathcal{B}$ is a σ-algebra. Furthermore $(A_1 \bigcup A_2) \subset (C_1 \bigcup C_2)$. $(C_1 \bigcup C_2) \in \mathcal{B}$, since $\mathcal{B}$ is a σ-algebra and $0 \leq \mu(C_1 \bigcup C_2) \leq \mu C_1 + \mu C_2 = 0$. So $\mu(C_1 \bigcup C_2) = 0$.

$\tilde{E}_1 = \sim(A_1 \bigcup B_1) = \tilde{A}_1 \bigcap \tilde{B}_1 = (C_1 \sim (A_1 \bigcup B_1)) \bigcup (\sim(B_1 \bigcup C_1))$. However $B_1 \bigcup C_1 \in \mathcal{B}$, and hence $\sim(B_1 \bigcup C_1) \in \mathcal{B}$. Also $C_1 \sim (A_1 \bigcup B_1) \subset C_1$. Hence $\tilde{E}_1$ has the appropriate form to be in $\mathcal{B}_0$.

Thus we need only show that the union of a countable collection of sets in $\mathcal{B}_0$ is in $\mathcal{B}_0$. Consider $\bigcup_{n=1}^{\infty} E_n,\ E_n \in \mathcal{B}_0 \Rightarrow \bigcup (A_n \bigcup B_n) = (\bigcup B_n) \bigcup (\bigcup A_n)$, we know that $\bigcup B_n \in \mathcal{B}$, so we only need that $\bigcup A_n$ is the subset of some set of measure 0. However $A_n \subset C_n$, with $\mu C_n = 0$, and the union of countable collection of things of meaurse 0 is still measure 0. Hence it suffices to consider the union of the C_n and we are done. Thus $\mathcal{B}_0$ is a σ-algebra.

Consider $E \in \mathcal{B}_0$ with $E = A \bigcup B$ as above so that A is a subset of a set of measure 0. We wish to show that μB is not dependant on the choice of $B \in \mathcal{B}$. Given two such representations $A_1 \bigcup B_1 = A_2 \bigcup B_2$. Let $A_1 \subset C_1$, with $\mu C_1 = 0$ and $A_2 \subset C_2$, with $\mu C_2 = 0$. Thus, $B_1 \supset \tilde{A}_1 \bigcap (A_2 \bigcup B_2) \Rightarrow \mu B_1 \geq \mu C_2 + \mu B_2 = \mu B_2$, however by symmetry of argument, we have $\mu B_2 \geq \mu C_1 + \mu B_1 = \mu B_1$, and hence $\mu B_1 = \mu B_2$.

Thus we may define $\mu_0 E = \mu B$, without concern that the value will change depending on our choice of B. So that $\mu_0 A = 0$ if $A \subset C, \mu C = 0$.

It then follows immediately that μ_0 is a measure from the fact that μ is a measure and that we have only added sets which have measure 0 and the sets generated by their unions with sets of $\mathcal{B}$, which will neccesarily preserve the additivity of the measure.

Thus we have created a complete measure with the neccesary properties.

Problem 11.10

- **Prove Proposition 7: Let *f* be a nonnegative measurable function . Then $\exists$ a sequence $<\varphi_n>$ of simple functions with $\varphi_{n+1} \geq \varphi_n$ such that $f = \lim \varphi_n$ at each point of *X*. If *f* is defined on a σ-finite measurable space, then we may choose the fucntions φ_n so that each vanishes outside a set of finite measure.**

- **Construction**

Define $E_{n,k} = \{x : k * 2^{-n} \leq f(x) < (k+1) * 2^{-n}\}$ and $\varphi_n = 2^{-n} \sum_{k=0}^{2^{2*n}} k * \chi_{E_{n,k}}$, with respect to each pair of integers $<n, k>$.

I wish to show that φ_n thus defined has the neccesary properties.

Since *f* is finite, for a given $<n, k>$, $n > 0$, $k > 0$, $mE_{n,k} < \infty$. Thus $m\left(\bigcup_{k=1}^{2^{2*n}} E_{n,k}\right) < \infty$, and so $\varphi_n =$ 0, except on that set which is finite. So each function vanishes off a set of finite measure.

Furthermore it should be clear that for a given n, the sets $E_{n,k}$ are mutually exclusive and that if for a given n and k, $f(x)$ is bracketed by $(k_0 * 2^{-n}, (k_0 + 1) * 2^{-n})$ then for the next n, we will have $f(x)$ bracketed by either $(2 * k_0 * 2^{-(n+1)}, (2 * k_0 + 1) * 2^{-n+1})$ or $((2 * k_0 + 1) * 2^{-(n+1)}, (2 * k_0 + 2) * 2^{-(n+1)})$. And the lower bound of the interval is the same as the value returned by the sum, it is then clear that $\varphi_{n+1} \geq \varphi_n$.

Furthermore, since the width of the interval between $k * 2^{-n}$ and $(k+1) * 2^{-n}$ goes to 0 as $n \to 0$, while the range goes to both 0 and ∞ as $k \to 0$ and $k \to 2^{2*n}$. This allows that for any value of $f(x)$, it will eventually be in some interval and the value can never exceed $f(x)$. Thus since the interval successively approximates $f(x)$ with an ϵ width equal to that of the interval hence the limit must go to $f(x)$.

Problem 11.17

- **Prove Proposition 15: If f and g are integrable functions and E is a measurable set, then**

  i. $\int_E (c_1 * f + c_2 * g) = c_1 * \int_E f + c_2 * \int_E g$
  ii. If $|h| \leq |f|$ and h is measurable then h is integrable.
  iii. If $f \geq g$ a.e., then $\int f \geq \int g$.

- **Proof**

  - **$\int_E (c_1 * f + c_2 * g) = c_1 * \int_E f + c_2 * \int_E g$**

    Simply break f and g into + and - parts and apply Proposition 13.

  - **If $|h| \leq |f|$ and h is measurable then h is integrable.**

    If we can first show part (iii), then by applying that result we can conclude that $-\infty < -\int |f| \leq \int h \leq \int |f| < \infty$, and thus h in integrable. So let us proceed to part (iii).

  - **If $f \geq g$ a.e., then $\int f \geq \int g$.**

    If we consider $\int f = \inf_{\varphi \leq f} \int \varphi$, then clearly $\exists$ a sequence of monotone increasing simple functions $< \varphi_n >$ which converge to g which will be less than f a.e. By montone convergence theorem $\int g = \lim \int \varphi_n \leq \inf_{\varphi \leq f} \int \varphi = \int f$. Hence $\int f \geq \int g$. QED.