# Real Analysis - Math 630

## Homework Set #12 - Chapter 12

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### Problem 12.2

- **Assume that $< E_i >$ is a sequence of disjoint measurable sets and $E = \bigcup E_i$. Then $\forall\ A$ we have that**

  **$\mu^*(A \bigcap E) = \sum \mu^*(A \bigcap E_i)$**

- **Proof**

If $< E_i >$ has only one non-empty set element then this is trivial. Suppose $< E_i >$ has a finite number of non-empty sets, then by the assumption of the induction hypothesis say that $\hat{E} = \bigcup_{i=1}^{n-1} E_i$, conforms to the property that $\mu^*\left(A \bigcap \hat{E}\right) = \sum_{i=1}^{n-1} \mu^*(A \bigcap E_i)$.

Consider $E = \bigcup_{i=1}^{n} E_i$, then $\mu^*(A \bigcap E) = \mu^*(A \bigcap E \bigcap E_n) + \mu^*(A \bigcap E \bigcap \tilde{E}_n)$, since $E_n$ is measurable. However $E \supset E_n$, and exploitning the fact that each $E_i$ is disjoint, we arrive at $\mu^*(A \bigcap E) = \mu^*(A \bigcap E_n) + \mu^*\left(A \bigcap \hat{E}\right) = \sum_{i=1}^{n} \mu^*(A \bigcap E_i)$. Thus the induction is proved. Hence we know for any finite sequence this property will hold.

So $\forall\ N$, $\mu^*(A \bigcap (\bigcup_{i=1}^{N} E_i)) = \sum_{i=1}^{N} \mu^*(A \bigcap E_i)$. However $A \bigcap (\bigcup_{i=1}^{N} E_i) \subset A \bigcap (\bigcup_{i=1}^{\infty} E_i)$, so $\mu^*(A \bigcap E) = \mu^*(A \bigcap (\bigcup_{i=1}^{\infty} E_i)) \geq \sum_{i=1}^{N} \mu^*(A \bigcap E_i)$. The left hand side is now independant of $N$ so we make take the right hand side to infinity. So $\mu^*(A \bigcap E) \geq \sum_{i=1}^{\infty} \mu^*(A \bigcap E_i)$, yet we know form the properties of outer measure that $\mu^*(A \bigcap E) \leq \sum_{i=1}^{\infty} \mu^*(A \bigcap E_i)$. Hence $\mu^*(A \bigcap E) = \sum \mu^*(A \bigcap E_i)$. QED.

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## Problem 12.4

- **Prove Proposition 9 - Let $C$ be a semialgebra of sets and $\mu$ a nonnegative set function defined on $C$ with $\mu\emptyset = 0$ (if $\emptyset \in C$). Then $\mu$ has a unique extension to a measure on the algebra $\mathcal{A}$ generated by $C$ if the following conditions are met:**

  - **i) If a set $C \in C$ is the union of a finite disjoint collection $\{C_i\}$ of sets in $C$, then $\mu C = \sum \mu C_i$**

  - **ii) If a set $C$ in $C$ is the union of a countable disjoint collection $\{C_i\}$ of sets in $C$, then $\mu C \leq \sum \mu \mathrm{C}_i$**

- **a) Condition (i) implies that if $A$ is the union of each of two finite disjoint collections $\{C_i\}$ and $\{D_i\}$ of sets in $C$, then $\sum \mu C_i = \sum \mu D_j$.**

- **Proof**

$\mu A = \sum \mu C_i$ by condition (i), but also by condition (i) we have that $\mu A = \sum \mu D_j$, so be the transitivity of equality, $\sum \mu C_i = \sum \mu D_j$.

- **b) Condition (ii) implies that $\mu$ is countably additive on $\mathcal{A}$.**

- **Proof**

By Problem 5, we have that all sets in $\mathcal{A}$ take the form of countable unions of sets in $C$. However it is always possible to generate a disjoint sequence of sets from an arbitrary sequence of sets, which preserve unions, provide that we may take finite intersections and complements, as we can on a semialgebra. Thus all elements of $\mathcal{A}$ may be expressed in terms of the union of a countable dijoint collection of sets.

This implies that condition (ii) is equivalent to $\forall\ A \in \mathcal{A}$, $\mu A \leq \sum \mu \mathrm{C}_i$, for any disjoint sequence $< C_i >$, such that $\bigcup C_i = A$. Which is exactly what is required to show that the measure is countably additive.

In conclusion, property (i) gives us that the notion of measure on $\mathcal{A}$ is well defined and unique on finite sets, and property (ii) gives us that countable additivity is preserved and that the value of $\mu A$ is still uniquely defined over countably additive unions.

## Problem 12.5

- **Let $\mathcal{C}$ be a semialgebra of sets and $\mathcal{A}$ the smallest algebra of sets containing $\mathcal{C}$.**

- **a) Show that $\mathcal{A}$ is comprised of sets of the form $A = \bigcup_{i=1}^{n} C_i$ with $C_i \in \mathcal{C}$.**

- **Proof**

Clearly it is neccesary that $\mathcal{A}$ be closed under finite unions and thus it will contain finite unions of elements in $\mathcal{C}$. We need to show that this is sufficient to be closed under finite intersections and complements.

Let $A, B \in \mathcal{A}$, such that $A = \bigcup_{i=1}^{n} A_i$, and $B = \bigcup_{i=1}^{m} B_i$, where each set $A_i$ and $B_i$ is in $\mathcal{C}$. Then $A \bigcap B = (\bigcup_{i=1}^{n} A_i) \bigcap (\bigcup_{i=1}^{m} B_i) = \bigcup_{i=1}^{n} (A_i \bigcap \bigcup_{j=1}^{m} B_j) = \bigcup_{i=1}^{n} \bigcup_{j=1}^{m} (A_i \bigcap B_j)$, but each $A_i \bigcap B_j \in \mathcal{C}$, since $\mathcal{C}$ is closed under finite intersections. Thus $A \bigcap B$ is a finite union and hence an element of $\mathcal{A}$ of the appropriate form. By induction $\mathcal{A}$ is closed under finite intersections.

Consider $\tilde{A} = \bigcap_{i=1}^{n} \tilde{C}_i$. However by definition of semialgebra, $\tilde{C}_i = \bigcup_{k=1}^{n_i} D_{i,k}$, with $D_{i,k} \in \mathcal{C}$. Thus $\tilde{A} = \bigcap_{i=1}^{n} \bigcup_{k=1}^{n_i} D_{i,k}$, but $\bigcup_{k=1}^{n_i} D_{i,k} \in \mathcal{A}$. Thus since $\mathcal{A}$ is closed under finite intersections from above, we must have that $\tilde{A} \in \mathcal{A}$.

Hence $\mathcal{A}$ is an algebra of sets. That it is the smallest algebra of sets containing $\mathcal{C}$ is clear from the fact that any algebra of sets must contain the elements of the form $\bigcup_{i=1}^{n} C_i$ with $C_i \in \mathcal{C}$, yet no other elements have been added besides these.

- **b) Show that $\mathcal{A}_\sigma = \mathcal{C}_\sigma$, so that $\mathcal{A}_\sigma$ and $\mathcal{A}_{\sigma\delta}$ may be replaced in theorems by $\mathcal{C}_\sigma$ and $\mathcal{C}_{\sigma\delta}$, respectively.**

- **Proof**

$\mathcal{A} \supset \mathcal{C}$, thus $\mathcal{A}_\sigma \supset \mathcal{C}_\sigma$. However $\mathcal{C}_\sigma$ contains all elements of the form $\bigcup_{i=1}^{\infty} C_i$, where $C_i \in \mathcal{C}$. Hence for each element in $\mathcal{A}$, $A = \bigcup_{i=1}^{n} C_i$, $A \subset \bigcup_{i=1}^{\infty} C_i$. Thus $\mathcal{A} \subset \mathcal{C}_\sigma \Rightarrow \mathcal{A}_\sigma \subset \mathcal{C}_{\sigma\sigma} = \mathcal{C}_\sigma$. Thus $\mathcal{A}_\sigma = \mathcal{C}_\sigma$. QED.

## Problem 12.6

- **Let $\mathcal{A}$ be a collection of sets which is closed under finite unions and finite intersections; an algebra of sets, for example.**

- **a) Show that $\mathcal{A}_\sigma$ is closed under countable unions and finite intersections.**

- **Proof**

Well by definition $\mathcal{A}_\sigma$ is the set of all sets generated by countable unions of sets in $\mathcal{A}$ so all that remains to be shown is that it is closed under finite intersections. Given $B \in \mathcal{A}_\sigma$ with $B \notin \mathcal{A}$, $C \in \mathcal{A}$. Then we want to show that $B \bigcap C \in \mathcal{A}_\sigma$. We know that there must exist a sequence of sets $< B_i >$, with each set in $\mathcal{A}$ such that $\bigcup B_i = B$, since it appears in the closure of $\mathcal{A}$ with respect to unions.

Define $< E_i >$ to be the sequence of sets such that $E_i = B_i \bigcap C \in \mathcal{A}$, since $\mathcal{A}$ is closed under finite intersections. However the $\bigcup E_i = \bigcup (B_i \bigcap C) = B \bigcap C$. Thus the intersection of any element of $\mathcal{A}_\sigma$ with one of $\mathcal{A}$ is in $\mathcal{A}_\sigma$. By induction we may intersect any finite number of elements in $\mathcal{A}$ with elements of $\mathcal{A}_\sigma$.

Given two elements $B, C \in \mathcal{A}_\sigma$, $B, C \notin \mathcal{A}$, then we know that $B \bigcap C = B \bigcap (\bigcup C_i)$, where $C_i$ are the elements of a generating sequence for $C$. This is equivalent to $\bigcup (B \bigcap C_i)$, however we know that $B \bigcap C_i \in \mathcal{A}_\sigma$ from above, and thus taking the countable union will also be in $\mathcal{A}_\sigma$. $\therefore$ $\mathcal{A}_\sigma$ is closed under finite intersections.

- **b) Show that each set in $\mathcal{A}_{\sigma\delta}$ is the intersection of a decreasing sequence of sets in $\mathcal{A}_\sigma$**

- **Construction**

Clearly if the set in question is a member of $\mathcal{A}_\sigma$ then the result is trivial, since we just take every term in the sequence to be that set.

Given $B \in \mathcal{A}_{\sigma\delta}$, $B \notin \mathcal{A}$, then $\exists$ a sequence $( < B_i >)_{i=0}^{\infty}$ of sets in $\mathcal{A}_\sigma$ such that $\bigcap B_i = B$.

Define $A_i = (\bigcup_{j=i}^{\infty} B_j) \bigcap (\bigcap_{k=0}^{i-1} B_k)$, with $A_0 = \bigcup B_i$. Given any element $x \in B$, $\Rightarrow$ $x \in B_k$, $\forall\, k$, hence $x \in A_i$, $\forall\, i$. Given $y \notin B$, $y \notin B_N$ for some N and hence $y \notin A_i$, $\forall\, i \geq N$. If $N$ is the least such N that works then $y \in A_i$, $\forall\, i < N$, and thus $A_i$'s form a decreasing sequence and $\bigcap A_i = B$. Thus we have a countable intersection of decreasing sets in $A_\sigma$ that yields $B \in A_{\sigma\delta}$.

## Problem 12.7

- **Let $\mu$ be a finite measure on an algebra $\mathcal{A}$, and $\mu^*$ the induced outer measure. Show that a set $E$ is measurable if and only if for each $\epsilon > 0$ there is a set $A \in \mathcal{A}_\delta$, $A \subset E$, such that $\mu^*(E \sim A) < \epsilon$.**
- **Proof**