Real Analysis - Math 630

Homework Set #13 - Chapter 12

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Problem 12.19

- **Let $X = Y$ be the set of positive integers, $\mathcal{A} = \mathcal{B} = \mathcal{P}(X)$, and let $\nu = \mu$ be the measure defined by setting $\mu(E)$ equal to the number of points in E if E is finite and ∞ if E is an infinite set. (This measure is called the counting measure.) State the Fubini and Tonelli Theorems explicitly for this case.**

- **Answer**

While I am unclear on exactly what suffices as response to this question, I am going to say when we can apply the Fubini and Tonelli theorems and then simpify the conclusions of the theorems as applicable to this specific case.

Let $f(x, y)$ be a real-valued function from $X \times Y$.

First considering the Tonelli theorem, we know that the set of positive integers is σ-finite, so we may apply the Tonelli theorem provided that f is nonnegative everywhere.

In the case of Fubini theorem we need that $\int_{X\times Y} f \, d(\mu\times\nu) < \infty$. However since $X\times Y$ is a set of cardinality $\aleph_0$ we know that $\exists$ an enumeration of the points in $X\times Y$. Let $< r_i >_{i=1}^{\infty}$ be such an enumeration. Then we can equate $f(x, y) = \sum_{i=1}^{\infty} f(r_i) * \chi_{\{r_i\}}(x, y)$. If we further define $\varphi_n(x, y) = \sum_{i=1}^{n} f(r_i) * \chi_{\{r_i\}}(x, y)$, then the sequence of functions are each simple and clearly converge to f. Suppose we consider the positive and negative parts of f. Then $f^{\pm} \geq \varphi_n^{\pm}$, and by application of the monotone convergence theorem, $\int f^{\pm} = \lim_{n\to\infty} \int \varphi_n^{\pm}$. However the right hand side is equivalent to $\lim_{n\to\infty} \sum_{i=1}^{n} f^{\pm}(r_i) * \mu(\{r_i\}) = \sum_{i=1}^{\infty} f^{\pm}(r_i)$, since the counting measure over a set of one element is 1. Thus saying that $\int_{X\times Y} f \, d(\mu\times\nu) < \infty$ is equivalent to asking that $\sum_{i=1}^{\infty} f^{+}(r_i) < \infty$ and $\sum_{i=1}^{\infty} f^{-}(r_i) < \infty$.

Let us now consider the implication of the theorems which are of course exactly the same. For simplicity I will not state the results generate by interchange of x and y.

- **(i) for almost all x the function f_x defined by $f_x(y) = f(x, y)$ is a measurable function on Y.**

  Well $f_x(y) =\sum_{i=1}^{\infty} f(r_i) * \chi_{\{r_i\}}(x, y)$, where x is fixed. Which is equal to $\sum_{i=1}^{\infty} f(x, i) * \chi_{\{i\}}(y)$.

  Consider $\{y : f_x(y) \geq \alpha\} = A_\alpha$. We need only show that μA_α is well defined for all α. However A_α is some subset of Y, and thus $A_\alpha \in \mathcal{P}(Y)$, so A_α is measurable, since it is in the collection of measurable sets, and hence $f_x(y)$ is a measurable function by Proposition 11.5.

- **(ii) $\int_Y f(x, y)\, d\nu(y)$ is a measurable function of X.**

  Let us explicitly compute $\int_Y f(x, y)\, d\nu(y) = \int_Y f_x(y)\, d\nu = \int_Y \sum_{i=1}^{\infty} f(x, i) * \chi_{\{i\}}(y)\, d\nu$. Using the same caveat as above we may consider this as positive and negative pieces of f. Since integration is a linear operation, we will only consider nonnegative functions f. In the nonnegative case we may exploit the fact that the sum is the limit of a sequence of increasing simple function and apply the monotone convergence theorem to say that

  $\int_Y f_x(y)\, d\nu = \sum_{i=1}^{\infty} f(x, i) * \nu(\{i\}) = \sum_{i=1}^{\infty} f(x, i)$, since $\nu(\{i\}) = 1,\ \forall\ i$.

  That the function $\sum_{i=1}^{\infty} f(x, i)$ is measurable follows immediately form the fact that $x \in X$, and $\mathcal{A} = \mathcal{P}(X)$. The argument is the same as the 2nd paragraph in (i).

- **(iii) $\int_X \int_Y f(x, y)\, d\nu\, d\mu = \int_{X \times Y} f(x, y)\, d(\mu \times \nu)$**

  Working from the conclusion of (ii), that $\int_Y f_x(y)\, d\nu = \sum_{i=1}^{\infty} f(x, i)$, we will then integrate with respect to μ, to find that $\int_X \int_Y f(x, y)\, d\nu\, d\mu = \sum_{i=1}^{\infty} \sum_{j=i}^{\infty} f(i, j)\, \mu(\{i\})$. (Note that technically we msut first consider this as the limit of finite sums, but the functions in the finite sums are increasing nonnegative functions whose limit is the ∞ sum, and the integrals are thus the same by Monotone convergence theorem.) However $\mu(\{i\}) = 1$ since μ is the counting measure, and thus $\int_X \int_Y f(x, y)\, d\nu\, d\mu = \sum_{i=1}^{\infty} \sum_{j=i}^{\infty} f(i, j)$. However we also know that $\int_{X \times Y} f(x, y)\, d(\mu \times \nu) = \int_{X \times Y} \sum_{i=1}^{\infty} f(r_i) * \chi_{\{r_i\}}(x, y)\, d(\mu \times \nu) = \sum_{i=1}^{\infty} f(r_i) * (\mu \times \nu)(\{r_i\}) = \sum_{i=1}^{\infty} f(r_i)$, since $(\mu \times \nu)(\{r_i\}) = 1,\ \forall\ i$. However since the r_i's range over ever pair (j, k) of integers exactly once, this is identical to the above result.

  Hence $\int_X \int_Y f(x, y)\, d\nu\, d\mu = \int_{X \times Y} f(x, y)\, d(\mu \times \nu) = \sum_{i=1}^{\infty} \sum_{j=i}^{\infty} f(i, j)$.

Problem 12.22

■ Let h and g be integrable functions on X and Y, and define $f(x, y) = h(x)\, g(y)$. Then f is integrable on $X \times Y$ and

$$\int_{X\times Y} f \, d(\mu\times\nu) = \int_X h \, d\mu * \int_Y g \, d\nu$$

■ Proof

Without loss of generality we may suppose that f, g and h are nonnegative functions. If not we may rewrite them as the differance of their negative and positive parts and by the linearity of integration we only need to show the proof for nonnegative functions.

Consider $\int_{X\times Y} f \, d(\mu\times\nu)$. If the integral exists then by definition of integration $\int_{X\times Y} f \, d(\mu\times\nu) = \inf\limits_{\varphi\geq f} \int_{X\times Y} \varphi \, d(\mu\times\nu) \quad = \sum_{i=1}^{n} c_i (\mu\times\nu)\,(E_i \bigcap (X\times Y))$, where $\varphi(x, y) = \sum_{i=1}^{n} c_i \, \chi_{E_i}(x, y)$ is a nonnegative simple function and each $E_i \subset X \times Y$.

However g, h are integrable functions, so they each admit a representation $\int_X h \, d\mu = \inf\limits_{\varphi_h \geq h} \int_X \varphi_h \, d\mu = \sum_{i=1}^{m_1} d_i \, \mu(H_i \bigcap X)$ and $\int_Y g \, d\nu = \inf\limits_{\varphi_g \geq g} \int_Y \varphi_g \, d\nu = \sum_{i=1}^{m_2} f_i \, \mu(G_i \bigcap Y)$, with $\varphi_h(x) = \sum_{i=1}^{m_1} d_i \, \chi_{H_i}(x)$ and $\varphi_g(y) = \sum_{i=1}^{m_2} f_i \, \chi_{G_i}(y)$.

Since $f(x, y) = h(x) * g(y)$ $\Rightarrow$ that every $\varphi_g * \varphi_h$ is a possible function for φ $\Rightarrow$ $\sum_{i=1}^{n} c_i \, \chi_{E_i}(x, y) = \sum_{i=1}^{m_1} d_i \, \chi_{H_i}(x) * \sum_{i=1}^{m_2} f_i \, \chi_{G_i}(y) \;=\; \sum_{i=1}^{m_1} \sum_{j=1}^{m_2} d_i \, f_j \, \chi_{H_i}(x)\, \chi_{G_j}(y)$. However $\chi_{H_i}(x)\, \chi_{G_j}(y) = \chi_{H_i \times G_j}(x, y)$.

This implies that f admits a representation as a sum of $n = m_1 * m_2$ elements, with each $E_i = H_{j(i)} \times G_{k(i)}$ and $c_i = d_{j(i)} * f_{k(i)}$, where the j, k depend on i and each pair of j, k are used exactly once over the $m_1 * m_2$ terms.

This that $\sum_{i=1}^{n} c_i (\mu\times\nu)\,(E_i \bigcap (X\times Y)) = \sum_{i=1}^{m_1 * m_2} d_{j(i)} * f_{k(i)} (\mu\times\nu)\,((H_{j(i)} \times G_{k(i)}) \bigcap (X\times Y))$. We know that each G_j and H_k is in X and Y respectively since these are the basis sets, so this devolves to $\sum_{i=1}^{m_1 * m_2} d_{j(i)} * f_{k(i)} (\mu\times\nu)\,(H_{j(i)} \times G_{k(i)})$. However $H_j \times G_k$ is a rectangle in $X \times Y$, so $(\mu\times\nu)\,(H_{j(i)} \times G_{k(i)}) = \mu(H_{j(i)}) * \nu(G_{k(i)})$

$\Rightarrow$ $\sum_{i=1}^{m_1 * m_2} d_{j(i)} * f_{k(i)} (\mu\times\nu)\,(H_{j(i)} \times G_{k(i)}) \;=\; \sum_{i=1}^{m_1 * m_2} d_{j(i)} * f_{k(i)} \, \mu(H_{j(i)}) * \nu(G_{k(i)})$, but by definition of the $j(i)$ and $k(i)$ this is just equal to $\sum_{i=1}^{m_1} d_i \, \mu(H_i) * \sum_{i=1}^{m_2} f_i \, \nu(G_i) = \int_X \varphi_h \, d\mu * \int_Y \varphi_g \, d\nu$.

Thus $\inf_{\varphi \geq f} \int_{X\times Y} \varphi \, d(\mu\times\nu) \leq \inf_{\varphi_h \geq h} \int_X \varphi_h \, d\mu * \inf_{\varphi_g \geq g} \int_Y \varphi_g \, d\nu$. Evaluating the infs we arrive at $\int_{X\times Y} f \, d(\mu\times\nu) \leq \int_X h \, d\mu * \int_Y g \, d\nu$, assuming the integral over f exists. However there is nothing specific in this argument to the fact we choose inf, except the last inequality, hence by the sup over simple functions less than the given function approach we can arrive at $\int_{X\times Y} f \, d(\mu\times\nu) \geq \int_X h \, d\mu * \int_Y g \, d\nu$, if the integral over f exists.

However what we have in fact shown is that $\int_X h \, d\mu * \int_Y g \, d\nu \leq \sup_{\psi \leq f} \int_{X\times Y} \psi \, d(\mu\times\nu) \leq \inf_{\varphi \geq f} \int_{X\times Y} \varphi \, d(\mu\times\nu) \leq \int_X h \, d\mu * \int_Y g \, d\nu$, the middle inequality is always true of the sups and infs used to define integrals. $\therefore$ $\int_X h \, d\mu * \int_Y g \, d\nu = \sup_{\psi \leq f} \int_{X\times Y} \psi \, d(\mu\times\nu) = \inf_{\varphi \geq f} \int_{X\times Y} \varphi \, d(\mu\times\nu)$, which implies that $\int_{X\times Y} f \, d(\mu\times\nu)$ exists and its value is $\int_X h \, d\mu * \int_Y g \, d\nu$. QED.

Author's note: I am all but sure that there is a better way to do this, but I don't know what it is so I decided to do it based on first principles.

Problem 12.24

- **The following example shows that we cannot remove the hypothesis that f be nonnegative from the Tonelli Theorem or that f be integrable from the Fubini Theorem. Let $X = Y$ be the positive integers and $\mu = \nu$ be the counting measure. Let**

$$f(x, y) = \begin{cases} 2 - 2^{-x}, & \text{if } x = y \\ -2 + 2^{-x}, & \text{if } x = y + 1 \\ 0, & \text{otherwise} \end{cases}$$

- **Contradiction**

First it is neccesary to notice that $f(x, y) = (2 - 2^{-x})\,\chi_A(x, y) + (-2 + 2^{-x})\,\chi_B(x, y)$, where $A = \{(x, y) : x = y\}$ and $B = \{(x, y) : x = y + 1\}$.

In order to disprove the Fubini and Tonelli thereoms it suffices to show that property (iii) in each fails to hold. In other words I wish to show that $\int_X \int_Y f(x, y)\, d\nu\, d\mu \neq \int_Y \int_X f(x, y)\, d\mu\, d\nu$.

Let us first consider $\int_X \int_Y f(x, y)\, d\nu\, d\mu$. So computing $\int_Y f(x, y)\, d\nu$ at a particular x, we have that $= (2 - 2^{-x})\,\nu(A_x) + (-2 + 2^{-x})\,\nu(B_x)$. Furthermore we know that for a fixed x, A_x has only one point determined by $x = y$, and similarly B_x has only one point. Hence the counting measure evaluated on these sets is just 1. Thus $\int_Y f(x, y)\, d\nu = (2 - 2^{-x}) + (-2 + 2^{-x}) = 0$. Hence $\int_X \int_Y f(x, y)\, d\nu\, d\mu = 0$.

Now let us consider $\int_Y \int_X f(x, y)\, d\mu\, d\nu \Rightarrow$ that we first evaluate $\int_X f(x, y)\, d\mu$ for a fixed y. Thus gives the integral as $(2 - 2^{-y})\,\mu(A_y) + (-2 + 2^{-y-1})\,\mu(B_y)$, note the substitution in the exponent in terms of the fixed coordinate. Again each of these has exactly one item in its set, with the exception of B_y at $y = 1$, which is empty because $x = 0$ is not in the domain of positive integers. So for $y \geq 2$, $\int_X f(x, y)\, d\mu = (2 - 2^{-y}) + (-2 + 2^{-y-1}) = 2^{-y}\left(\frac{1}{2} - 1\right) = -2^{-y-1}$. Also at $y = 1$, $\int_X f(x, y)\, d\mu = (2 - 2^{-1}) = \frac{3}{2}$.

Now in order to compute the integral over Y, it is not hard to see that by restricing the integral to the first N integers we are in fact integrating over a simple function. Thus $\int_Y \int_X f(x, y)\, d\mu\, d\nu = \int_{\{1,2,\ldots,N\}} \left(\frac{3}{2}\,\chi_{\{1\}} + -2^{-y+1} * \chi_{\mathbb{N}\sim\{1\}}\right) d\nu = \frac{3}{2}\,\nu(\{1\}) + \sum_{i=2}^{N} -2^{-i-1} * \nu(\{i\})$ Since we are using the counting measure, each evaluation is in fact one. Simplifying the geometric series we have

$\frac{=3}{2} - \frac{\frac{1}{2}}{1-\frac{1}{2}} = \frac{1}{2}$. Thus $\int_Y \int_X f(x, y)\, d\mu\, d\nu = \frac{1}{2} \neq 0 = \int_X \int_Y f(x, y)\, d\nu\, d\mu$. Hence neither the Fubini nor the Tonelli theoreoms will hold for this particular fucntion and choice of measure. QED.

Problem 12.25

- **The following example shows that we cannot remove the hypothesis that f be integrable from the Fubini Theorem or that μ and ν are σ-finite from the Tonelli Theorem: Let $X = Y$ be the interval [0, 1], with $\mathcal{A} = \mathcal{B}$ the class of Borel sets. Let μ be the Lebesgue measure and the ν the counting measure. Then the diagonal $\Delta = \{ < x, y > \in X \times Y : x = y\}$ is measurable (is an $\mathcal{R}_{\sigma\delta}$, in fact), but its characteristic function fails to satisfy any of the equalities in condition (iii) of the Fubini and Tonelli Theorems.**

- **Contradiction**

Again we will seek an exception of the Fubini and Tonelli Theorems by computing the differant halves of part (iii) in each result and showing that they are not equal.

Consider $\int_X \int_Y \chi_\Delta(x, y)\, d\nu\, d\mu$. First we wish to compute $\int_Y \chi_\Delta(x, y)\, d\nu$, for a particular choice of x. This means that $\int_Y \chi_\Delta(x, y)\, d\nu = \nu(\Delta_x)$, but $\chi_{\Delta_x}(y)$ is only non-zero for the one value where $x = y$, and so since ν is the counting measure, we have that $\int_Y \chi_\Delta(x, y)\, d\nu = 1$. So $\int_X \int_Y \chi_\Delta(x, y)\, d\nu\, d\mu = \int_X 1\, d\mu = \mu(X) = 1$.

Now consider $\int_Y \int_X \chi_\Delta(x, y)\, d\mu\, d\nu$. First we start with $\int_X \chi_\Delta(x, y)\, d\mu$, for a particular value of y. Which evaluates the integral to $\mu(\Delta_y)$, which again is a set containing only the point where $x = y$. However since μ is the Lesbegue measure this evaluates to 0. So we have that $\int_Y \int_X \chi_\Delta(x, y)\, d\mu\, d\nu = \int_Y 0\, d\nu = 0$. Since $0 \neq 1$, we have thus shown that χ_Δ fails to satisfy the (iii) property of the Fubini and Tunelli Theorems. QED.

Problem 12.31

- **Let f be a nonnegative measurable function on $(-\infty, \infty)$, and let m_2 be the two-dimensional Lebesgue measure on $\mathbb{R}^2$. Then**

  $$\boldsymbol{m_2\{<x, y> : 0 \leq y \leq f(x)\} = m_2\{<x, y> : 0 < y < f(x)\} = \int f(x)\, dx}$$

  Let $\varphi(t) = m\,\{x : f(x) \geq t\}$. Then φ is a decreasing function and

  $$\boldsymbol{\int_0^\infty \varphi(t)\, dt = \int f(x)\, dx.}$$

- **Proof**

- **$m_2\{<x, y> : 0 \leq y \leq f(x)\} = m_2\{<x, y> : 0 < y < f(x)\} = \int f(x)\, dx$**

Essentially in the first part we are attempting to show that $\int f(x)\, dx$ is the same thing as the area under the curve (with and without its boundary).

Let $A = \{<x, y> : 0 \leq y \leq f(x)\}$ and $B = \{<x, y> : 0 < y < f(x)\}$. Let μ denote the Lesbegue measure. Consider $\int \chi_A\, d(\mu \times \mu)$, since $\mathbb{R}$ is σ-finite and χ is everywhere nonnegative, we may apply Tonelli Theorem to conclude that $\int \chi_A\, d(\mu \times \mu) = \int\int \chi_A\, d\mu\, d\mu = \int \mu(A_x)\, d\mu$. For any particular x we have that A_x is the interval $[0, f(x)]$, and thus that $\mu(A_x) = f(x)$. Thus $\int \chi_A\, d(\mu \times \mu)$ $= \int f(x)\, d\mu$, but the integral with respect to Lesbegue measure is exactly what is intended by the notation $\int f(x)\, dx$. Finally we know that $\int \chi_A\, d(\mu \times \mu) = (\mu \times \mu)(A)$ by the definition of an integral of a characteristic function, but $(\mu \times \mu) = m_2$. Hence we have that $m_2\, A = \int f(x)\, dx$.

If we then note that B behaves exactly the same as A in the above proof since $B_x = (0, f(x))$, and hence $\mu(B_x) = \mu(0, f(x)) = f(x)$. Thus the conclusion is equally valid that $m_2\, B = \int f(x)\, dx$.

- $\int_0^\infty \varphi(t)\,dt = \int f(x)\,dx$

  While in the above proof we made a construction that showed the area was the same as that under the curve $f(x)$, by exploiting a series of vertical bars, in the proof that follows we will attempt to demonstrate that it is equivalent to a series of horizontal bars.

  On the question that φ is a decreasing function, this follows immediately from that fact that $mA \le mB$, whenever $A \subseteq B$, and the observation that if $x \in \{x : f(x) \ge t_1\}$ then $x \in \{x : f(x) \ge t_2\}$, $\forall$ $t_2 \le t_1$. Thus for $t_1 > t_2$, we have that $\{x : f(x) \ge t_1\} \supseteq \{x : f(x) \ge t_2\}$. So $m\{x : f(x) \ge t_1\} \ge m\{x : f(x) \ge t_2\}$.

  Let $\Delta = \{ < x, t > \; : f(x) \ge t \ge 0\}$

  Consider $\int_0^\infty \varphi(t)\,dt = \int_0^\infty m\{x : f(x) \ge t\}\,dt = \int_0^\infty m(\Delta_t)\,dt = \int_0^\infty \int_{-\infty}^\infty \chi_\Delta(x, t)\,dx\,dt$. The last equality coming from the fact that the measure of a cross-section is the same as the integral of its characteristic function. Since characteristic functions are bounded and both measure are taken with respect to the σ-finite real numbers, we may apply Tonelli thoerem to conclude that $\int_0^\infty \int_{-\infty}^\infty \chi_\Delta(x, t)\,dx\,dt = \int_{-\infty}^\infty \int_0^\infty \chi_\Delta(x, t)\,dt\,dx$. From here we need only observe that Δ is the same as A in the first section of this problem, upto the labelling of variables. Thus from above we may conclude that $\int_{-\infty}^\infty \int_0^\infty \chi_\Delta(x, t)\,dt\,dx = \int f(x)\,dx$. $\therefore \int_0^\infty \varphi(t)\,dt = \int f(x)\,dx$. QED.