Real Analysis - Math 630

Homework Set #3 - Chapter 3

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9-21-00

Problem 18

- **Show that (v) does not imply (iv) in Propostion 18 by constructing a function *f* such that $\{x : f(x) > 0\} = E$, a given non-measurable set, and such that f assumes each value at most once.**

- **Counter-Example**

Let E = the standard non-measurable set on [0, 1]

Let $f(x) = \begin{cases} x, & x \in E \\ -x, & x \notin E \end{cases}$ defined on domain $[0, \infty)$

Thus $\{x : f(x) > 0\} = E$, and $f(x) = \alpha$ has at most one solution for any α, namely α or $-\alpha$.

Hence $\{x : f(x) = \alpha\}$ is measurable $\forall\ \alpha$ but $\{x : f(x) > 0\}$ is not measurable.

Problem 19

- **Let D be a dense set of real numbers. Let f be an extended real-valued function on $\mathbb{R}$ such that $\{x : f(x) > \alpha\}$ is measurable $\forall\ \alpha \in D$. Show that f is measurable.**

- **Proof**

We wish to show that $\{x : f(x) > \alpha\}$ is measurable $\forall\ \alpha \in D$ is equivalent to the condition $\{x : f(x) > \alpha\}$ is measurable $\forall\ \alpha$.

Take $\beta \notin D$ then define $A_n = \{x : f(x) > \delta_n$, with $\delta_n \in (\beta\text{-}\frac{1}{n}, \beta) \bigcap D\}$.

We know that $(\beta\text{-}\frac{1}{n}, \beta) \bigcap D$ is non-empty $\forall\ n$ since, D is dense.

Consider $\bigcap_{n=1}^{\infty} A_n \subset \bigcap_{n=1}^{\infty} \left\{x :\ f(x) > \beta - \frac{1}{n}\right\} = \{x : f(x) \geq \beta\}$. But $\{x : f(x) \geq \beta\} \subset \bigcap_{n=1}^{\infty} A_n$ since $\forall\ n$, $\delta_n < \beta$. Thus $\bigcap_{n=1}^{\infty} A_n = \{x : f(x) \geq \beta\}$ which means that $\{x : f(x) \geq \beta\}$ is measurable and hence by Proposition 18, $\{x : f(x) > \beta\}$ is measurable, so $\{x : f(x) > \alpha\}$ is measurable $\forall\ \alpha$. QED

Problem 20

- **Show that the sum and product of two simple functions are simple.**

- **Proof**

- $\boldsymbol{\chi_{A \bigcap B} = \chi_A * \chi_B}$

$$\chi_{A \bigcap B} = \begin{cases} 1, & \text{if } x \in A \bigcap B \\ 0, & \text{if } x \notin A \bigcap B \end{cases}$$

$$\chi_A * \chi_B = \begin{cases} 1, & \text{if } x \in A \text{ and } x \in B \\ 0, & \text{if } x \notin A \text{ or } x \notin B \end{cases}$$

But x ∈ A and x ∈ B ⇔ x ∈ A ⋂ B and, x∉A or x∉ B ⇔ x ∉ A ⋂ B, thus $\chi_{A \bigcap B} = \chi_A * \chi_B$, QED.

- $\chi_{A \bigcup B} = \chi_A + \chi_B - \chi_A * \chi_B$

  $\chi_{A \bigcup B} = \begin{cases} 1, & \text{if } x \in A \bigcup B \\ 0, & \text{if } x \notin A \bigcup B \end{cases}$

  $\chi_A + \chi_B - \chi_A * \chi_B = \chi_A + \chi_B - \chi_{A \bigcap B} = \begin{cases} 1, & \text{if } x \in A \text{ or } x \in B \\ 0, & \text{if } x \notin A \text{ and } x \notin B \end{cases}$

  But x $\in$ A or x $\in$ B $\Leftrightarrow$ x $\in$ A $\bigcup$ B and, x$\notin$A and x$\notin$B $\Leftrightarrow$ x $\notin$ A $\bigcup$ B, thus $\chi_{A \bigcup B} = \chi_A + \chi_B - \chi_A * \chi_B$. QED.

- $\tilde{\chi_A} = 1 - \chi_A$

  $\tilde{\chi_A} = \begin{cases} 0, & \text{if } x \in A \\ 1, & \text{if } x \notin A \end{cases}$

  $1 - \chi_A = \begin{cases} 1 - 1 = 0, & \text{if } x \in A \\ 1 - 0 = 1, & \text{if } x \notin A \end{cases}$

  Thus $\tilde{\chi_A} = 1 - \chi_A$. QED

Problem 21

- **a) Let *D* and *E* be measurable sets and *f* a function with domain $D \bigcup E$. Show that *f* is measurable if and only if its restrictions to *D* and *E* are measurable.**

- **Proof**

  f is measurable $\Leftrightarrow \{x : f(x) > \alpha, \ x \in D \bigcup E\}$ is measurable $\forall \alpha$.

  $A \equiv \{x : f(x) > \alpha, \ x \in D \bigcup E\} = B \equiv \{x : f(x) > \alpha, \ x \in D\} \bigcup C \equiv \{x : f(x) > \alpha, \ x \in E\}$

  Clearly if B and C are measurable then A is measurable as the union of measurable sets.

  Since D is measurable, we have that $A \sim D$ is measurable if A is measurable, but $A \sim D = C$, so C would be measurable, and similarly $A \sim E = B$ gives B measurable. Thus if A is measurable B and C are measurable.

  Since B and C are equivalent to f restricted to D and E respectively, we must have that f is measurable if and only if its restrictions to D and E are measurable. QED.

- **b) Let f be a function with measurable domain D. Show that f is measurable iff the function g defined by $g(x) = f(x)$ for $x \in D$ and $g(x) = 0$ for $x \notin D$ is measurable.**

- **Proof**

f is measurable $\Leftrightarrow \{x : f(x) > \alpha,\ x \in D\}$ is measurable $\forall\ \alpha$.

$\forall\ \alpha \geq 0$, $\{x : g(x) > \alpha\} = \{x : f(x) > \alpha,\ x \in D\}$, thus $\{x : g(x) > \alpha\}$ is measurable iff f is measurable.

$\forall\ \alpha < 0$, $\{x : g(x) > \alpha\} = \{x : f(x) > \alpha,\ x \in D\} \bigcup \{x : x \notin D\}$.

However the first term on the right is clearly measurable iff f is measurable, and the second is just x $\in \tilde{D}$, but $\tilde{D}$ is measurable since D is measurable. Thus $\{x : g(x) > \alpha\}$ is measurable for $\alpha < 0$ iff f is measurable.

$\therefore$ $\{x : g(x) > \alpha\}$ is measurable $\forall\ \alpha$ iff f is measurable, so g is measurable iff f is measurable. QED

Problem 23

- **Prove Proposition 23 by establishing the following lemmas:**

- **a) Given a measurable function f on [a, b] that takes the values $\pm\infty$ only on a set of measure 0, and given $\epsilon > 0$, $\exists\ M$ such that $|f| \leq M$ except on a set of measure less than $\epsilon / 3$.**

- **Proof**

We know that A = $\{x : f(x) \geq -\alpha\}$ and B = $\{x : f(x) \leq \alpha\}$ are measurable for all α.

Thus C_α = A $\bigcap$ B is measurable = $\{x : |f(x)| \leq \alpha\}$.

We also know that $\tilde{C_\alpha} \supset \tilde{C_{\alpha+1}}$, since $\alpha < \alpha + 1$.

Furthermore we know that since f is infinite on only a set of measure 0, then $\exists\ \alpha_0$ such that $m\tilde{C_{\alpha_0}} < \infty$. Thus by Proposition 14, we have that $m \bigcap_{n=1}^{\infty} \tilde{C_{\alpha_0+n}} = \lim_{n->\infty} m\,\tilde{C}_{\alpha_0+n} = 0$. Hence by definition of limit, $\exists\ D = \tilde{C_{\alpha_0+n}}$, for some n, with $mD < \epsilon / 3$. QED.

■ b) Let f be a measurable function on [a, b]. Given $\epsilon > 0$ and M, ∃ a simple function φ such that $|f(x) - \varphi(x)| < \epsilon$ except where $|f(x)| \geq M$. If $m \leq f \leq M$, then we may take φ so that $m \leq \varphi \leq M$.

■ Proof

Let φ be a simple function with values of n*(ϵ / 2) where n ∈ $\mathbb{Z} \cap (-(M+1)*2/\epsilon, (M+1)*2/\epsilon)$. Thus simply choose the value for n that satisfies $|f(x) - \varphi(x)| < \epsilon$ when $|f(x)| < M \Leftrightarrow -\epsilon - f(x) < -\varphi(x) < -f(x) + \epsilon \Rightarrow \epsilon + f(x) > \varphi(x) > f(x) - \epsilon \Rightarrow \epsilon + f(x) > n*(\epsilon / 2) > f(x) - \epsilon \Rightarrow 1 + f(x) / \epsilon > n / 2 > f(x) / \epsilon - 1 \Rightarrow 2 + 2 * f(x) / \epsilon > n > 2 * f(x) / \epsilon - 2$.

Such an n being guaranteed to exist by the fact that ∃ at least one integer in every range 1 + a > x > a - 1. Thus we have defined φ such that it has the neccesary property. This construction will clearly yield results of the form $m \leq \varphi \leq M$, when $m \leq f \leq M$, except possibly within ϵ / 2 of the bounds where we may choose the bound itself to satisfy our criterion.

■ c) Given a simple function φ on [a, b], ∃ a step function g on [a, b] such that $g(x) = \varphi(x)$ except on a set of measure less than ϵ / 3. If $m \leq \varphi \leq M$ then we can take g such that $m \leq g \leq M$.

■ Proof

Let $\delta = \frac{b-a}{2^n}$, define $[a_m, b_m] = [a + (m-1)*\delta, a + m*\delta]$, with $m \in \mathbb{Z} \cap [0, 2^n]$. Thus $[a_m, b_m]$ partition [a, b] into n intervals of width δ.

We know that $\varphi(x) = \sum_{i=1}^{N} \beta_i * \chi_{A_i}$, with $A_i = \{x : \varphi(x) = \beta_i\}$.

Define $g(x) = \varphi(\xi_m)$, $\forall\ x \in [a_m, b_m)$, with $\xi_m = \beta_i$ such that $m(A_i \cap [a_m, b_m))$ is the max over all i.

Clearly g(x) is a step function and in the limit n → ∞ we have that $g(x) = \varphi(x)$. Also we have that $m(\{x : g(x) = \varphi(x)\}) = \sum_{m=1}^{2^n} m(A_i \cap [a_m, b_m))$, for the A_i associated with ξ_m.

Hence $K_n = m(\{x : g(x) \neq \varphi(x)\}) = \sum_{m=1}^{2^n} m(\tilde{A}_i \cap [a_m, b_m))$.

However K_n forms a strictly decreasing series since for successive n, each pair of new partition pieces must cover at least as well as the original. And we know that limit $K_n \to 0$ so, ∃ n such that $K_n < \epsilon / 3\ \forall\ \epsilon > 0$. Thus ∃ some step function g conforming to the same bounds as φ that will have $g(x) = \varphi(x)$ except on a set of measure less than ϵ / 3.

d) Given a step function g on [a, b], ∃ a continuous function h on [a, b] such that h(x) = g(x) except on a set of measure less than ϵ / 3. If $m \leq g \leq M$ then we can take h such that $m \leq h \leq M$.

Proof

Clearly for any interval [c, d], we may construct a continuous curve joining g(c) and g(d).
Since g is a step function, ∃ a partition $x_0 < x_1 < ... < x_N$ such that for each interval (x_n, x_{n+1}), g(x) assumes only one value. Let $v_n(x)$ be any continuous function joining g($x_n - \frac{\epsilon}{6*(N+1)}$) and g($x_n + \frac{\epsilon}{6*(N+1)}$), which does not exceed its boundary points, for n ∈ ℤ ⋂ [1, N-1].

Define h(x) = $\begin{cases} g(x), \text{ if } \left| x - x_n \right| \frac{> \epsilon}{6*(N+1)}, \forall n \\ v_n(x), \text{ if } \left| x - x_n \right| \frac{\leq \epsilon}{6*(N+1)} \end{cases}$

Thus h(x) is neccesarily a continuous function and h(x) = g(x) except for N intervals of length $2 * \frac{\epsilon}{6*(N+1)}$, which has total measure $\frac{\epsilon * N}{3*(N+1)} < \frac{\epsilon}{3}$. Thus by explicit construction we have found aa continuous h satisfying the conditions. QED.

Problem 29

Give an example to show that we must require $mE < \infty$ in Proposition 23.

Example

Let E = (-∞, ∞). Let $f_n(x) = \frac{x}{n}$. Thus $f_n(x) \to 0$.

Take $\epsilon > 0$. Then $|f_n(x) - 0| = |f_n(x)| < \epsilon$ ∀ n ≥ some N implies that $|x| <$ N*ϵ.

Hence B = (-N*ϵ, N*ϵ) is the collection of all x such that $|f_n(x)| < \epsilon$ ∀ n ≥ N. Thus $\tilde{B} \equiv$ A = (-∞, -N*ϵ] ⋃ [N*ϵ, ∞) is the smallest set such that ∀ x ∈ $\tilde{A}$ and ∀ n ≥ N, $|f_n(x)| < \epsilon$. This is true because if we were to remove any element of A, we would add an element to $\tilde{A}$ with $|f_n(x)| \geq \epsilon$. Hence the measure of any set, K, with the neccesary property of Proposition 23 will be larger than $mA = \infty$. Thus ∀ $\delta > 0$ we have $mK = \infty > \delta$. Hence showing a counter-example.

Problem 30

- **Prove Ergoroff's Theorem: If <f_n> is a sequence of measurable functions that converge to a real-valued function f a.e. on a measurable set E of finite measure, then given $\eta > 0$, $\exists$ a subset $A \subset E$ with $mA < \eta$ such that f_n converges to f uniformly on $E \sim A$.**

- **Proof**

Define $A_i \subset E$ for $i \in \mathbb{N}$ with $mA_i < \delta_i = 2^{-i}\,\eta$ and a smallest N_i such that $\forall\ x \notin A_i$, and all $n \geq N_i$, $|f_n(x) - f(x)| < \epsilon_i = \frac{1}{i}$. We know that A_i exists by Proposition 24.

Let $A = \bigcup A_i$. Thus $mA \leq \sum mA_i < \sum 2^{-i}\,\eta = \eta$.

Since $\tilde{A} = \sim \bigcup A_m = \bigcap \tilde{A_m} \Rightarrow \forall$ x $\in E \sim A$ that f_n converges to f since $|f_n(x) - f(x)| < \epsilon_m \to 0$.

Thus we have f_n converging with some N_m depending only on ϵ_m, when f_n(x) restricted to A. So it demonstrates uniform convergence. QED