Real Analysis - Math 630

Homework Set #4 - Chapter 4

by Bobby Rohde
9-28-00

Problem 1

■ a) Show that if

$$f(x) = \begin{cases} 0, & x \text{ is irrational} \\ 1, & x \text{ is rational} \end{cases}$$

Then

$$R\overline{\int_a^b} f(x)\,dx = b - a \text{ and } R\underline{\int_a^b} f(x)\,dx = 0$$

■ Proof

$R\overline{\int_a^b} f(x)\,dx = \inf \int_a^b \psi(x)\,dx$, ∀ step functions $\psi(x) \geq f(x)$, ∀ x.
But each step function is composed of a finite collection of open intervals, and in the domain each open interal ∃ an rational number since the rationals are dense. Thus $\psi(x) \geq f(x)$, ∀ x implies that the value over any open interval of the step function $\psi(x)$ is ≥ 1. Thus the infimum of the integral of all such step functions must be when $\psi(x) = 1$, ∀ x. ∴
$R\overline{\int_a^b} f(x)\,dx = \inf \int_a^b \psi(x)\,dx = (b-a) * 1 = b - a$.

$R\underline{\int_a^b} f(x)\,dx = \sup \int_a^b \phi(x)\,dx$, ∀ step functions $\phi(x) \leq f(x)$, ∀ x. Since irrationals are dense, each interval in the composition of ϕ has an irrational in its domain. Thus $\phi(x) \leq f(x)$, ∀ $x \Rightarrow$ that over any open interval in ϕ, $\phi(x) \leq 0$. Thus the supremum of the integral of all such step functions must be when $\phi(x) = 0$. Hence $R\underline{\int_a^b} f(x)\,dx = \sup \int_a^b \phi(x)\,dx = 0$. QED

■ b) Construct a sequence <f_n> of nonnegative, Riemann integrable functions such that f_n increases monotonically to f. What does this imply about changing the order of integration and the limiting process.

■ Construction

Let $f_n = x^{\frac{1}{n}}$ over (0,1], so limit n → ∞ we have $f = 1$. $\int_0^1 f_n \, dx = \frac{n+1}{n}$. So $\lim_{n\to\infty} \int_0^1 f_n \, dx = 1 = \int_0^1 f \, dx = \int_0^1 \lim_{n\to\infty} f_n \, dx$.

Thus we have that the limit of the integral is the integral of the limit.

Problem 2

- **a) Let f be a bounded function on $[a, b]$, and let h be the upper envelope of f. Then $R\,\overline{\int}_a^b f = \int_a^b h$.**

- **Proof**

$R\,\overline{\int}_a^b f(x)\,dx = \inf \int_a^b \psi(x)\,dx$, $\forall$ step functions $\psi(x) \geq f(x)$, $\forall\, x$.
We know from Problem 2.51(a) that h(x) ≥ f(x), and from 2.51(b) we have that h(x) is upper semi-continuous so that h(y) ≥ $\overline{\lim}_{x\to y}\, h(x) = \inf\limits_{\delta>0} \sup\limits_{0<|x-y|<\delta} h(x)$.

From 2.51(a), $h(x) = f(x)$ if and only if f is upper semi-continuous at x, so $\psi(x) \geq h(x)$, except possibly where $f(x)$ is not upper semi-continuous $\Rightarrow$ $f(y) < \overline{\lim}_{x\to y}\, f(x)$. However $h(y) = \inf\limits_{\delta>0} \sup\limits_{|x-y|<\delta} f(x)$ so h(y) = $\overline{\lim}_{x\to y}\, f(x)$, when $f(y) < \overline{\lim}_{x\to y}\, f(x)$. Let A = {x : $\psi(x) < h(x)$}

Suppose A ≠ $\emptyset$. Let $\gamma \in$ A. So h(γ) > $\psi(\gamma)$, and h(γ) = $\overline{\lim}_{x\to\gamma}\, f(x)$. Let $\epsilon = h(\gamma)$-$\psi(\gamma)$. By definition of lim sup we have that $\forall\, \delta > 0$ such that | sup f(x) - $h(\gamma)$ | < ϵ / 2 and 0 < | x - γ | < δ, $\exists\, \alpha$ in that range such that $|(f(\alpha) + \epsilon/2) - h(\gamma)| < \epsilon/2 \Rightarrow |(-f(\alpha) - \epsilon/2) + h(\gamma)| < \epsilon/2 \Rightarrow -f(\alpha) + h(\gamma) < \epsilon \Rightarrow -f(\alpha) + h(\gamma) < h(\gamma) - \psi(\gamma) \Rightarrow f(\alpha) > \psi(\gamma)$. Which means that α and γ must be in differant intervals of the partition of ψ, however this works $\forall\, \delta > 0$ which implies that γ must be a boundary point between intervals of the partition. However since ψ is a step function it is composed of a finite number of distinct intervals and thus has a finite number of boundary points. So A has at most a finite number of items in it.

Thus $\psi \geq h$ except on a set of measure 0, so $R\,\overline{\int}_a^b f(x)\,dx = \inf \int_a^b \psi(x)\,dx \geq \int_a^b h(x)\,dx$.

Define <$\phi_n(x)$> to be a sequence of step function such that ϕ_n is over a equipartition of [a, b] into 2^n pieces, with the value of ϕ_n over each piece to be sup $h(x)$ over that piece.

BWOC Suppose $\lim_{n\to\infty} \phi_n(x) \neq h(x)$. That implies that $\exists\, \epsilon > 0$ such that $\forall\, n$, | $\phi_n(x)$ - $h(x)$ | ≥ $\epsilon \Rightarrow$ $\forall$ partition intervals including x, $\exists\, \gamma$ in that interval with h(γ) ≥ h(x) + ϵ. However h(x) ≥ $\overline{\lim}_{\xi\to x}\, h(\xi)$, which gurantees that for some small distance $\zeta > 0$, h(x) + ϵ > h(ξ) $\forall\, \xi \in$ (x-ζ, x+ζ). Thus we have a contradiction since h(γ) must be < h(x) + ϵ for γ within ζ of x.

Thus $\lim_{n\to\infty} \phi_n(x) = h(x)$ and clearly ϕ_n is bounded since h and f are bounded. Thus by

Proposition 6 we have that $\int_a^b h\, dx = \lim_{n\to\infty} \int_a^b \phi_n\, dx$. Furthermore ϕ_n are each step functions ≥ f so that $\lim_{n\to\infty} \int_a^b \phi_n\, dx \geq \inf \int_a^b \phi\, dx = R\,\overline{\int}_a^b f(x)\, dx$, where ϕ is any step function ≥ f.

Hence $\int_a^b h\, dx \geq R\,\overline{\int}_a^b f(x)\, dx$. So using the previous bound we have $R\,\overline{\int}_a^b f = \int_a^b h$. QED.

■ b) Use Part (a) to prove that: A bounded function *f* on $[a, b]$ is Riemann integrable if and only if the set of points at which *f* is discontinuous has measure 0.

■ Proof

A bounded function *f* is Riemann integrable if and only if $R\,\overline{\int}_a^b f = R\,\underline{\int}_a^b f$.

Since *f* is bounded we know $R\,\overline{\int}_a^b f = \int_a^b h$, where h is the upper envelope from (a). By symmetry of defintion, interchanging the supremums and infimums we know that $R\,\underline{\int}_a^b f = \int_a^b g$, where g is the lower envelope.

So *f* is Riemann integrable if and only if $\int_a^b h = \int_a^b g$. However 2.51(c) gives us that $g(x) = h(x)$ if and only *f* is continuous at x. So by Proposition 5(i), we know that $\int_a^b h = \int_a^b g \Leftrightarrow \int_a^b (h - g) = 0$. But since $h(x) \geq g(x)$, ∀ x we also know that $\int_a^b (h - g) = 0$ if and only if, the set of points were $h(x) \neq g(x)$ has measure 0, since otherwise ∃ some $\alpha > 0$ such that $m(\{x : h(x) - g(x) = \alpha\}) > 0$, and hence $\int_a^b (h - g) \geq \alpha *$(measure of that set) > 0. But the set of points $h(x) \neq g(x)$ is identical to the set of the discontinuities in *f*. $\therefore$ *f* is Riemann integrable if and only if the set of points at which *f* is discontinuous has measure 0. QED.

Prove parts (iii) - (v) of Proposition 5

■ (iii) If $f \leq g$ a.e., then

$$\int_E f \;\leq\; \int_E g$$

Hence

$$\left|\int_E f\,\right| \;\leq\; \int_E (|f\,|)$$

■ Proof

$\int_E f \;\leq\; \int_E g \Leftrightarrow \int_E f \;-\; \int_E g \;\leq\; 0 \Leftrightarrow \int_E (f-g) \;\leq\; 0$. However $f \;\leq\; g$ a.e. $\Leftrightarrow f-g \leq\; 0$ a.e.

Thus for simple functions $\phi \leq f-g \leq 0$ a.e., sup $\int_E \phi \;\leq\; 0$. Since $\phi \leq 0$ except on a set of measure 0. However we know that the integral must exist since the integral of f and g exist. ∴ since it exists it must equal sup $\int_E \phi$, so $\int_E (f-g) \;\leq\; 0 \Leftrightarrow \int_E f \;\leq\; \int_E g$. QED.

It follows immediately that $\int_E (|f\,|) \;\geq\; \int_E f$, since $|f\,| \geq f$. But also $|f\,| = |-f\,| \geq -f$, so $\int_E (|f\,|) \;\geq\; \int_E -f$ $\Longrightarrow -\int_E (|f\,|) \;\leq\; \int_E f, \Rightarrow -\int_E (|f\,|) \;\leq\; \int_E f \;\leq\; \int_E (|\,f\,|) \Rightarrow \left|\int_E f\,\right| \;\leq\; \int_E (|f\,|)$. QED

■ (iv) If $A \leq f(x) \leq B$, then

$$A * m\,E \leq \int_E f \;\leq\; B * mE$$

■ Proof

Since $A \leq f(x) \leq B$, we know from (iii) that $\int_E A \;\leq\; \int_E f \;\leq\; \int_E B$. However the integral of a constant is just that constant times the measure of the set integrated over, so $\int_E A \;=\; A * mE$ and $\int_E B \;=\; B * mE$. Hence $A * m\,E \leq \int_E f \;\leq\; B * mE$. QED.

■ (v) If A and B are disjoint measurable sets of finte measure, then

$$\int_{A\cup B} f \;=\; \int_A f \;+\; \int_B f$$

■ Proof

$\int_{A\cup B} f \;=\; \int f * \chi_{A\cup B} \;=\; \int f * (\chi_A \;+\; \chi_B)$, since $\chi_{A\cup B} = \chi_A \;+\; \chi_B$.
However from (i) we have that $\int f * (\chi_A \;+\; \chi_B) = \int f * \chi_A \;+ \int f * \chi_B \;= \int_A f \;+\; \int_B f$