Real Analysis - Math 630

Homework Set #5 - Chapter 4

by Bobby Rohde
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Problem 3

- **Let f be a nonnegative measurable function. Show that $\int f = 0$ implies that $f = 0$ a.e.**

- **Proof**

BWOC, suppose $f \neq 0$ a.e. Then $\exists$ some set E with $mE > 0$ such that $f(x) > 0$, $\forall\, x \in E$.

Since $\{x \mid f(x) > 0\} = \bigcup_{n \in \mathbb{N}} \{x \mid f(x) \geq \frac{1}{n}\}$, we know that the countable sum of the measures of the sets on the righthand side is greater than or equal to the measure of the left hand side which is non-zero. So $\exists\, N$ such that if $A = \{x \mid f(x) \geq \frac{1}{N}\}$, then $mA > 0$.

Define $h(x) = \min\{f(x), \frac{1}{N}\}$, $\forall$ x. Thus $h(x) \leq f(x)$, $\forall$ x. Let $F \subseteq A$, such that $0 < mF < \infty$.

We know $\int f = \sup_{g \leq f} \int g$, and $\int f \geq \int_F f$, the latter coming from the fact that since f is nonnegative and $\int_F f = \int f * \chi_F$ with $f * \chi_F \leq f$, $\forall$ x. This implies $\int f \geq \int_F f = \sup_{g \leq f} \int_F g \geq \int_F h \geq \frac{1}{N} * m F > 0$. Thus $\int f > 0$, which contradicts the assumption that $\int f = 0$, which implies that $f = 0$ a.e. QED

Problem 4

■ Let f be a nonnegative measurable function.

■ **a) Show that ∃ an increasing sequence $<\varphi_n>$ of nonnegative simple functions each of which vanishes outside a set of finite measure such that $f = \lim \varphi_n$.**

■ **Construction**

Define $\varphi_n = \begin{cases} \frac{m-1}{2^n}, \ \forall \ x \in [-n, n] \text{ such that } \frac{m-1}{2^n} \le f(x) < \frac{m}{2^n} \text{ and } \forall \ m \in \mathbb{N} \text{ with } m \le 2^{2n} \\ 0, \text{ otherwise} \end{cases}$

Thus φ_n assumes exactly 2^{2n}-1 non-zero values over a set of measure $\le 2n$. Thus φ_n is a nonnegative simple function and vanishes except on a set of finite measure.

Furthermore we can show that $\varphi_{n+1} \ge \varphi_n$ ∀ x and n as follows. Suppose $\varphi_n(x) = 0$ then clearly $\varphi_{n+1}(x) \ge 0$, thus suppose $\varphi_n(x) = \frac{m-1}{2^n}$ for some integer m with $0 < m \le 2^{2n}$. This implies that $\frac{m-1}{2^n} \le f(x) < \frac{m}{2^n} \Rightarrow \frac{2(m-1)}{2^{n+1}} = \frac{m-1}{2^n} \le f(x) < \frac{m}{2^n} = \frac{2m}{2^{n+1}}$, but if $m-1 \le 2^{2n}$, then clearly $2(m-1) \le 2 * 2^{2n} < 2^{2(n+1)}$, so thus we have that $\varphi_{n+1}(x) \ge \frac{2(m-1)}{2^{n+1}} = \frac{m-1}{2^n} = \varphi_n(x)$, so its an increasing sequence.

Now we need only show that $f = \lim \varphi_n$.

Let [x] denote the least natural number ≥ x, and $\log_2 x$ = to the logarithim base 2 of x, with $\log_2 0$ = -∞.

Given $\epsilon > 0$ and some x_0, we may choose $N = \max\{[|x_0|], [-\log_2 \epsilon], [\log_2 f(x_0)]\}$.
We wish to show that ∀ n > N, $|\varphi_n(x_0) - f(x_0)| < \epsilon$.

Since $n > [|x_0|] \ge |x_0|$, we have that $x_0 \in [-n, n]$ so x_0 meets the first condition for assigning a non-zero value to $\varphi_n(x_0)$.

Since $n > [\log_2 f(x_0)] > \log_2 f(x_0)$, we know that $2^n > 2^{\log_2 f(x_0)} = f(x_0)$. So $f(x_0) < \frac{m}{2^n} \le \frac{2^{2n}}{2^n} = 2^n$, for some m ∈ ℕ, m $\le 2^{2n}$. Thus x_0 meets the second condition for assigning a non-zero value to $\varphi_n(x_0)$.

Finally since $n > [-\log_2 \epsilon]$ and n meets the requirements to assign x_0 a particular non-zero value, we know that $|\varphi_n(x) - f(x)| \le$ to the spacing of the values of $\varphi_n = \frac{1}{2^n} < \frac{1}{2^{[-\log_2 \epsilon]}} \le \frac{1}{\epsilon^{-1}} = \epsilon$. Thus $|\varphi_n(x) - f(x)| < \epsilon$. ∴ φ_n converges pointwise to f ∀ x. So $f = \lim \varphi_n$.

Thus we have constructed a φ_n satisfying all required properties, QED.

- **b) Show that $\int f = \sup \int \varphi$ over all simple functions $\varphi \leq f$.**

- **Proof**

By definition $\int f = \sup_{h \leq f} \int h$, where h is a bounded measurable function with $m\{x : h(x) \neq 0\}$ is finite.

We may of course limit ourselves to taking the sup over all $h'(x)$ such that $h'(x) = \max\{h(x), 0\}$. However all such $h'(x)$ are nonnegative measurable functions. The measurability of $h'(x)$ deriving from the fact that $\{x : h(x) \geq 0\}$ and the set $\{x : h(x) < 0\}$ must both be measurable.

Hence we may apply part a) and conclude that $\forall\ h'(x)$, $\exists$ an increasing sequence $< \varphi_n >$ such that $\varphi_n \rightarrow h'(x)$. Thus by Proposition 10 we have that $\int h' = \lim \int \varphi_n$ which equals the sup $\int \varphi_n$ since it is an increasing sequence. Thus the sup over some subset of all such simple functions φ must be = $\sup_{h \leq f} \int h = \int f$. So $\int f \leq \sup \int \varphi$.

However $\varphi \leq f$, and since we are considering a sup we may assume that $\varphi \geq 0$. Thus by Proposition 8 we have that $\int \varphi \leq \int f$. So $\int f \geq \sup \int \varphi$.

$\therefore \int f = \sup \int \varphi$, over all simple functions $\varphi \leq f$. QED

Problem 5

- **Let f be a nonnegative integrable function. Show that the function F defined by**

$$F(x) = \int_{-\infty}^{x} f$$

is continuous by using Theorem 10.

- **Proof**

In order to show continuity we need that $F(a) = \lim_{x\to a} F(x)$, $\forall\, x$. $\Rightarrow$ We need $\int_{-\infty}^{a} f = \lim_{x\to a} \int_{-\infty}^{x} f$. Define $f_n = \begin{cases} f(x), & x < a - \frac{1}{n} \\ 0, & \text{otherwise} \end{cases}$ then f_n is an increasing sequence with $f_n \to f$ as n→∞ ∀ x ∈ (-∞, a).

$\forall\, x < a$, we have that $\int_{-\infty}^{x} f < \int_{-\infty}^{a} f_N$ for some N such that $x < a - \frac{1}{N} < a \Rightarrow N > \frac{1}{a-x}$.

Thus $\lim_{x\to a^-} \int_{-\infty}^{x} f \leq \lim_{n\to\infty} \int_{-\infty}^{a} f_n$, and by Theorem 10 we have that $\lim_{n\to\infty} \int_{-\infty}^{a} f_n = \int_{-\infty}^{a} f$. But also we know that $\forall\, x$ such that $0 < a - x < \delta < 1$, we have that $\int_{-\infty}^{x} f > \int_{-\infty}^{a} f_M$, $\forall\, M < \frac{1}{\delta}$, thus as $\delta \to 0$ in the limit $x \to a^-$, we get that $\lim_{x\to a^-} \int_{-\infty}^{x} f \geq \lim_{n\to\infty} \int_{-\infty}^{a} f_n$.

So $\lim_{x\to a^-} \int_{-\infty}^{x} f = \lim_{n\to\infty} \int_{-\infty}^{a} f_n = \int_{-\infty}^{a} f$.

Now consider $\lim_{x\to a^+} \int_{-\infty}^{x} f = \int_{-\infty}^{a} f + \lim_{x\to a^+} \int_{a}^{x} f$, by Proposition 12. We know that since f is integrable $f < \infty$ a.e. Since if $f = \infty$ on some set, A, of measure > 0, we would have $\int f \geq mA * \infty = \infty$, which is false since $\int f < \infty$, by definition of integrable. Thus without loss of generality we may assume that $f < \infty$ everywhere since this won't change it's integral. Now since f is finite over a closed interval we know that f is bounded over that interval. Let $N(x)$ be the maximum of $f(y)$ ∀ y ∈ [a, x]. Hence, $\int_{a}^{x} f \leq m$[a, x]*$N(x)$, by Proposition 5. However it is clear that $N(x_1) \leq N(x_2)$, if $a < x_1 \leq x_2$. Thus $\lim_{x\to a^+} \int_{a}^{x} f \leq \lim_{x\to a^+} m[a,\ x] * N(x) = 0$.

Thus $\lim_{x\to a^+} \int_{-\infty}^{x} f = \int_{-\infty}^{a} f + \lim_{x\to a^+} \int_{a}^{x} f = \int_{-\infty}^{a} f$. Hence $\int_{-\infty}^{a} f = \lim_{x\to a} \int_{-\infty}^{x} f \Rightarrow F(a) = \lim_{x\to a} F(x)$. $\therefore$ F is continuous.

Problem 6

- **Let $< f_n >$ be a sequence of nonnegative measurable functions that converge to f, and suppose $f_n \leq f \; \forall$ n. Then**

$$\int f = \lim \int f_n$$

- **Proof**

By Fatou's Lemma we know that $\int f \leq \underline{\lim} \int f_n$. Also since $f_n \leq f \; \forall$ n, we know that $\int f_n \leq \int f$, $\forall$ n, and thus $\overline{\lim} \int f_n \leq \int f$.

However $\underline{\lim} \int f_n \leq \overline{\lim} \int f_n$ by definition of lim sup and lim inf. So by squeezing, $\overline{\lim} \int f_n = \int f = \underline{\lim} \int f_n = \lim \int f_n$. QED

Problem 7

- **a) Show that we may have strict inequality in Fatou's Lemma.**

- **Proof**

Consider the sequence $< f_n >$ defined by $f_n(x) = 1$ if $n \leq x < n + 1$, with $f_n(x) = 0$ otherwise.

$\forall x, \exists$ N > x, such that $f_n(x) = 0, \forall$ n > N. Thus $f_n \to f = 0$. So $\int f = 0$, however $\underline{\lim} \int f_n = 1$, since $\forall$ n, we have $\int f_n = 0 + \int_n^{n+1} 1 = 1$. Thus $0 < 1 \Rightarrow \int f < \underline{\lim} \int f_n$. QED

- **b) Show that the Monotone Convergence Theorem need not hold for decreasing sequences of functions.**

- **Proof**

Let $f_n(x) = 0$ if $x < n$ and $f_n(x) = 1$ for $x \geq n$.

$\forall x, \exists$ N > x, such that $f_n(x) = 0, \forall$ n > N. Thus $f_n \to f = 0$. Hence $\int f = 0$. However consider $\lim \int f_n$. $\int f_n \geq \int_n^{n+1} f_n = 1$. Hence $\lim \int f_n \geq 1 \neq 0$. Hence $\lim \int f_n \neq \int f$. QED