Real Analysis - Math 630

Homework Set #6 - Chapter 4

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Problem 10

- **a) Show that if f is integrable over E then so is $|f|$ and**

$$\left|\int_E f\right| \leq \int_E |f|$$

Does the integrability of $|f|$ imply integrability of f?

- **Proof**

f is integrable $\Rightarrow$ f^+ and f^- are each integrable. However $|f| = f^+ + f^-$, so by Proposition 15ii, we have that $|f|$ is integrable and $\int_E |f| = \int_E f^+ + \int_E f^-$. Also $\left|\int_E f\right| = \left|\int_E f^+ - \int_E f^-\right| \leq \left|\int_E f^+\right| + \left|\int_E f^-\right|$, but since f^+ and f^- are nonnegative we can drop the asbsolute value signs, so $\left|\int_E f\right| \leq \int_E f^+ + \int_E f^- = \int_E |f|$. So we have shown the first part.

$|f| = f^+ + f^-$, So $|f|$ integrable $\Rightarrow$ $(f^+ + f^-)$ is integrable. However the set $A = \{x \mid f^+(x) \neq 0\}$ is disjoint from the set $B = \{x \mid f^-(x) \neq 0\}$, and A and B must be measurable iff f is a measurable function. Thus by Proposition 15, $\int_E (f^+ + f^-) = \int_{A \cup B} (f^+ + f^-) = \int_A f^+ + \int_B f^-$. Thus f^+ and f^- are each integrable iff f is a measurable function.

But this implies that $\int_A f^+ - \int_B f^- = \int_{A \cup B} (f^+ - f^-) = \int_E f$, is integrable. Thus $|f|$ implies integrability of f iff f is a measurable function.

b) The improper Riemann integral of a function may exist without the function being integrable. If f is integrable, show that the improper Riemann integral is equal to the Lebesgue integral whenever the former exists.

Proof

Let $R\int_a^b f\,dx$, be a Riemann integrable function with improper limit point a, and f defined for all x $\in$ (a, b]. Note that if any other point in (a, b] is improper, we may into two or more integrals each of which with one improper limit point so we need only assume one limit is improper. In particular we know from analysis that improper points must be disconnected in order for a function to be Riemann integrable. WLOG I assume that a < b, but all parts of the proof may be reversed for a > b, and will still hold accordingly.

Case |a| < ∞

Then consider the sequence of measurable integrable functions $<f_n>$, such that $f_n(x) = \begin{cases} f(x), & a + 1/n \leq x \leq b \\ 0, & \text{otherwise} \end{cases}$. Then $\lim f_n(x) = f(x)$ a.e. Also we have that $|f_n| \leq f$. Thus by Theorem 16 we have that $\int_a^b f = \lim \int_a^b f_n$. But by construction we must have that $f_n(x)$ is bounded, which allows us to apply Proposition 4 to get that $\int_a^b f_n = R\int_a^b f_n$, and hence $\lim \int_a^b f_n = \lim R\int_a^b f_n = R\int_a^b \lim f_n = R\int_a^b f$. So $\int_a^b f = R\int_a^b f$.

Case a = -∞

Then consider the sequence of measurable integrable functions $<f_n>$, such that $f_n(x) = \begin{cases} f(x), & -n \leq x \leq b \\ 0, & \text{otherwise} \end{cases}$. Then $\lim f_n(x) = f(x)$ a.e. Also we have that $|f_n| \leq f$. Thus by Theorem 16 we have that $\int_a^b f = \lim \int_a^b f_n$. But by construction we must have that $f_n(x)$ is bounded, which allows us to apply Proposition 4 to get that $\int_a^b f_n = R\int_a^b f_n$, and hence $\lim \int_a^b f_n = \lim R\int_a^b f_n = R\int_a^b \lim f_n = R\int_a^b f$. So $\int_a^b f = R\int_a^b f$. QED

Problem 11

- **If φ is a simple function, we have two definitions for $\int \varphi$, $\int \varphi = \int \varphi^+ - \int \varphi^-$ and $\int \varphi = \sum_{i=1}^n a_i \, mA_i$. Show that they are equal.**

- **Proof**

Consider $\int \varphi = \int \varphi^+ - \int \varphi^-$, By problem 4b, we know that $\forall$ nonnegative measurable functions f, $\int f = \sup \int \psi$, with the sup taken over all simple functions $\leq f$, and the right hand side evaluated according to the old rule for integrating step functions. Since it follows immediately from the definition of f^+ and f^- that φ^+ and φ^- are simple, we can thus deduce that $\int \varphi^+ = \sup \int \psi = \sum_{i=1}^{n_1} b_i \, mB_i$, $\int \varphi^- = \sup \int \psi = \sum_{i=1}^{n_2} c_i \, m\mathrm{C}_i$, where $\varphi^+ = \sum_{i=1}^{n_1} b_i \, \chi_{B_i}$ and $\varphi^- = \sum_{i=1}^{n_2} c_i \, \chi_{C_i}$. So $\int \varphi = \int \varphi^+ - \int \varphi^- = \sum_{i=1}^{n_1} b_i \, mB_i - \sum_{i=1}^{n_2} c_i \, m\mathrm{C}_i$, also we know that $B_i \cap C_j = \emptyset \; \forall \; i, \; j$ and each b_i corresponds to some $a_j > 0$ and each c_i corresponds to some $a_j < 0$, so $\sum_{i=1}^{n_1} b_i \, mB_i - \sum_{i=1}^{n_2} c_i \, m\mathrm{C}_i = \sum_{i=1}^{n_1} b_i \, mB_i + \sum_{i=1}^{n_2} (-c_i) \, m\mathrm{C}_i = \sum_{i=1}^{n} a_i \, mA_i$. Since the sets C_i and B_i, must follow the same correspondence. QED.

Problem 12

- **Let g be an integrable function on a set E and suppose that $< f_n >$ is a sequence of measurable functions such that $|f_n(x)| \leq g(x)$ a.e. on E. Then**

$$\int_E \underline{\lim} \, f_n \leq \underline{\lim} \int_E f_n \leq \overline{\lim} \int_E f_n \leq \int_E \overline{\lim} \, f_n$$

- **Proof**

We know for free that $\underline{\lim} \int_E f_n \leq \overline{\lim} \int_E f_n$, since this is true for any lim sups and infs.

We only need to show that $\int_E \underline{\lim} \, f_n \leq \underline{\lim} \int_E f_n$ and $\overline{\lim} \int_E f_n \leq \int_E \overline{\lim} \, f_n$.

f_n is measurable and bounded by an integrable function, therefore f_n is integrable. Also we know that $\underline{\lim} \, f_n \leq f_n \; \forall \; n$ thus By Proposition 15iii, we have that $\int_E \underline{\lim} \, f_n \leq \int_E f_n \; \forall \; n \Rightarrow \int_E \underline{\lim} \, f_n \leq \underline{\lim} \int_E f_n$. A similar argument for lim sup $f_n \geq f_n$ gives us that $\overline{\lim} \int_E f_n \leq \int_E \overline{\lim} \, f_n$, and thus we are done. QED

Problem 15

■ Let f be integrable over E. Then given $\epsilon > 0$,

- **a) ∃ a simple function φ such that**

$$\int_E |f - \varphi| < \epsilon$$

- **Construction**

Since $\int_E |f - \varphi| \geq |\int_E f - \varphi| = |\int_E f - \int_E \varphi| = |\int_E f^+ - \int_E f^- - \int_E \varphi|$, it suffices to show that the latter is less than ϵ. If we further specify that $A = \{x \mid f^+(x) \neq 0\}$ and $B = \{x \mid f^-(x) \neq 0\}$, then we may clearly choose $\varphi = 0$, $\forall$ x $\in E \sim (A \cup B)$. So we may further simplify the above expression to $|\int_E f^+ - \int_E f^- - \int_E \varphi| = |\int_E f^+ - \int_E f^- - \int_A \varphi - \int_B \varphi| \geq |\int_E f^+ - \int_A \varphi| - |\int_E f^- + \int_B \varphi|$, which again all that we need to show is that this new expression is less than ϵ.

By Problem 4, we know that $\forall$ nonnegative measurable functions f we have that $\int f = \sup \int \varphi$, taken over all simple functions $\varphi \leq f$. Since f^+ and f^- are nonnegative measurable functions we know there must exist simple functions φ_S and φ_I respectively such that $0 \leq \int f^+ - \int \varphi_S < \epsilon/2$ and $0 \leq \int f^- - \int \varphi_I < \epsilon/2$. The second implies that $0 \leq \int f^- + \int(-\varphi_I) < \epsilon/2$. Thus if we choose φ over $A = \varphi_S$ and φ over $B = -\varphi_I$, then we have that $|\int_E f^+ - \int_A \varphi| - |\int_E f^- + \int_B \varphi| < \epsilon/2 + \epsilon/2 = \epsilon$. Thus we have shown the construction as required.

- **a) ∃ a step function ψ such that**

$$\int_E |f - \psi| < \epsilon$$

- **Construction**

By the Question 3.23c, which part of the proof to Proposition 22, we know that ∃ a step function ψ such that ψ = a simple function φ except on a set of less than $\epsilon / 3$ and if M bounds φ then M bounds ψ, $\forall\ \epsilon > 0$. Let $\epsilon > 0$ and φ be a step function such that $\int_E |f - \varphi| < \epsilon/2$. Let N be a number such that $|\varphi(x)| < N$, $\forall\ x$. This is possible since simple functions are always bounded. Choose ψ such that $\psi = \varphi$ except on a set of measure $\frac{\epsilon}{4*N}$ with the set denoted by A, and ψ bounded by N. Thus $\int_E |f - \psi| = \int_{E\sim A} |f - \varphi| + \int_A |f - \psi| < \int_{E\sim A} |f - \varphi| + \int_A |f - \psi| = \int_{E\sim A} |f - \varphi| + \int_A |f - \varphi + \varphi - \psi| \leq \int_{E\sim A} |f - \varphi| + \int_A (|f - \varphi| + |\varphi - \psi|) \leq \int_{E\sim A} |f - \varphi| + \int_A |f - \varphi| + \int_A |\varphi - \psi| \leq \epsilon / 2 + 2*N*mA = \epsilon / 2 + 2*N*\frac{\epsilon}{4*N} = \epsilon$.

Therefore $\int_E |f - \psi| < \epsilon$.

- **a) ∃ a continuous function g such that**

$$\int_E |f - g| < \epsilon$$

- **Construction**

By question 3.23d, we have that ∃ a continuous function g such that g = a step function ψ except on a set of measure less than $\epsilon / 3$, which agress with the bounds on ψ. With this we may replace the occurances of φ in the above proof with ψ and ψ with g and the proof will precede identically, except that we have to infer the boundedness of ψ from the boundedness of the preceding φ. Thus we may construct g such that $\int_E |f - g| < \epsilon$.