Real Analysis - Math 630

Homework Set #7 - Chapter 5

by Bobby Rohde
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Problem 1

- **Let f be the function defined by $f(0) = 0$ and $f(x) = x * \sin(\frac{1}{x})$ for $x \neq 0$. Find $D^+ f(0)$, $D_+ f(0)$, $D^- f(0)$, and $D_- f(0)$.**

- **$D^+ f(0)$**

  $D^+ f(0) = \overline{\lim}_{h \to 0^+} \frac{f(h)-f(0)}{h} = \overline{\lim}_{h \to 0^+} \frac{h * \sin(\frac{1}{h})}{h} = \overline{\lim}_{h \to 0^+} \sin(\frac{1}{h}) = 1$, since for arbitrarily small positive h we can find a value of $\frac{1}{h}$ which gives sine its maximal value.

- **$D_+ f(0)$**

  $D_+ f(0) = \underline{\lim}_{h \to 0^+} \frac{f(h)-f(0)}{h} = \underline{\lim}_{h \to 0^+} \frac{h * \sin(\frac{1}{h})}{h} = \underline{\lim}_{h \to 0^+} \sin(\frac{1}{h}) = -1$, since for arbitrarily small positive h we can find a value of $\frac{1}{h}$ which gives sine its minimal value.

- **$D^- f(0)$**

  $D^- f(0) = \overline{\lim}_{h \to 0^+} \frac{f(0)-f(-h)}{h} = \overline{\lim}_{h \to 0^+} \frac{-h * \sin(\frac{1}{-h})}{h} = \overline{\lim}_{h \to 0^+} \sin(\frac{1}{h}) = 1$, since for arbitrarily small positive h we can find a value of $\frac{1}{h}$ which gives sine its maximal value.

- **$D_- f(0)$**

  $D_- f(0) = \underline{\lim}_{h \to 0^+} \frac{f(0)-f(-h)}{h} = \underline{\lim}_{h \to 0^+} \frac{-h * \sin(\frac{1}{-h})}{h} = \underline{\lim}_{h \to 0^+} \sin(\frac{1}{h}) = -1$, since for arbitrarily small positive h we can find a value of $\frac{1}{h}$ which gives sine its minimal value.

Problem 2

a) Show that $D^+[-f(x)] = -D_+ f(x)$

Proof

$$D^+[-f(x)] = \overline{\lim_{h\to 0^+}} \frac{-f(x+h)-[-f(x)]}{h} = \overline{\lim_{h\to 0^+}} - \frac{f(x+h)-f(x)}{h} = \inf_{h\to 0^+} \sup_{0<\epsilon\leq h} \left(-\frac{f(x+\epsilon)-f(x)}{\epsilon}\right) =$$
$$\inf_{h\to 0^+} \left(-\inf_{0<\epsilon\leq h} \frac{f(x+\epsilon)-f(x)}{\epsilon}\right) = -\left(\sup_{h\to 0^+} \inf_{0<\epsilon\leq h} \frac{f(x+\epsilon)-f(x)}{\epsilon}\right) = -\underline{\lim_{h\to 0^+}} \frac{f(x+h)-f(x)}{h} = -D_+ f(x).$$ QED.

b) If $g(x) = f(-x)$, then $D^+ g(x) = -D_- f(-x)$

Proof

$$-D_- f(-x) = -\underline{\lim_{h\to 0^+}} \frac{f(-x)-f(-x-h)}{h} = -\underline{\lim_{h\to 0^+}} \frac{g(x)-g(x+h)}{h} = \overline{\lim_{h\to 0^+}} - \frac{g(x)-g(x+h)}{h} = \overline{\lim_{h\to 0^+}} g(x+h) - \frac{g(x)}{h} =$$
$$D^+ g(x).$$ QED

Problem 3

■ a) If f is continuous on [a, b] and assummes a local maximum at $c \in (a, b)$, then

$$D_+ f(c) \leq D^+ f(c) \leq 0 \leq D_- f(c) \leq D^- f(c)$$

■ Proof

Well $D^+ f(c) = \overline{\lim_{h\to 0^+}} \frac{f(c+h)-f(c)}{h} \geq D_+ f(c) = \underline{\lim_{h\to 0^+}} \frac{f(c+h)-f(c)}{h}$, since lim sups are neccesarily greater than lim infs. Similarly $D_- f(c) \leq D^- f(c)$. So we only really need to establish that $D^+ f(c) \leq 0 \leq D_- f(c)$. Since $f(c)$ is a local maximum $\exists$ an open interval I $\subset [a, b]$ such that c $\in$ I and $f(c)$ is the maximum of f over I. WLOG we may consider h to be sufficiently small such that $c + h$ and $c - h \in$ I.

Consider $D^+ f(c) = \overline{\lim_{h\to 0^+}} \frac{f(c+h)-f(c)}{h}$, since $c + h \in$ I, $f(c+h) < f(c) \Rightarrow f(c+h) - f(c) < 0$. Furthermore since $h > 0$, we know that $\frac{f(c+h)-f(c)}{h} < 0$. By definition $\overline{\lim_{h\to 0^+}} g(h) = \inf_{h>0} \sup_{0<\xi<h} g(\xi)$, but as h → 0 , the supremums form a nonincreasing sequence, and hence so long as we permit ± ∞ as acceptable limits, $\overline{\lim_{h\to 0^+}} g(h)$ must exist.

Thus we have that $\overline{\lim_{h\to 0^+}} \frac{f(c+h)-f(c)}{h} \leq 0$, since each $\frac{f(c+h)-f(c)}{h} < 0$, and the limit exists.

Similarly $D_- f(c) = \underline{\lim_{h\to 0^+}} \frac{f(c)-f(c-h)}{h}$ has $\frac{f(c)-f(c-h)}{h} > 0$ over I, and the limit must exist over the extended reals, so $\underline{\lim_{h\to 0^+}} \frac{f(c)-f(c-h)}{h} \geq 0$. Hence $D^+ f(c) \leq 0 \leq D_- f(c)$. QED.

■ b) What if f has a local maximum at a or b?

■ Answer

If f is maximal at a or b then the two derivates which are still inside the interval must exist and obey the above inequality, however the other two may not even exist. Thus with $a < b$ and $f(a)$ a local maximum we have that $D_+ f(a) \leq D^+ f(a) \leq 0$, and if $f(b)$ is a local max then $0 \leq D_- f(b) \leq D^- f(b)$.

Problem 4

- **Prove: If f is continuous on $[a,\ b]$ and one of its derivates (say D^+) is everywhere nonnegative on $(a,\ b)$, then f is nondecreasing on $[a,\ b]$; i.e. $f(x)\ \leq\ f(y)$ for $x\ \leq\ y$.**

- **Proof**

  - **Lemma - For a function $g(x)$ such that $D^+\,g(x) \geq \epsilon > 0$, the above property holds.**

    Suppose $\exists\ x < y$, such that $g(x) > g(y)$. Since g is continuous we may apply the normal knowledge of continuous functions to conclude that either g has a local maximum value or g is always decreasing. If g were always decreasing then we know that $D^+\,g(x) = \overline{\lim_{h\to 0^+}} \frac{g(x+h)-g(x)}{h} < 0$ since $g(x+h) - g(x) < 0, \forall$ x. Which is a contradiction, thus g has a local max.

    However by problem 3, $D^+\,g(c)\ \leq\ 0$, where c is the location of the local max, but $D^+\,g(x)\ > 0, \forall$ x which gives a contradiction. Thus we know that g is nondecreasing.

  - **Consider $f(x)\ =\ g(x)\ -\ \epsilon * x$, where $g(x)$ is defined as in the lemma.**

    Therefore $D^+\,f(x)\ =\ \overline{\lim_{h\to 0^+}} \frac{f(x+h)-f(x)}{h} = \overline{\lim_{h\to 0^+}} \frac{g(x+h)-\epsilon*(x+h)-(g(x)-\epsilon*x)}{h} = \overline{\lim_{h\to 0^+}} \left(\frac{g(x+h)-g(x)}{h} - \frac{\epsilon*h}{h}\right) = \overline{\lim_{h\to 0^+}} \left(\frac{g(x+h)-g(x)}{h} - \epsilon\right) = \overline{\lim_{h\to 0^+}} \left(\frac{g(x+h)-g(x)}{h}\right) - \epsilon \geq 0$.

    So if f is an arbitrary function of the type given in the problem then we know that $\forall\ \epsilon > 0$, $f(x) + \epsilon * x$, is a nondecreasing function. Suppose $\exists\ x < y$ such that $f(x) > f(y)$, yet we know that $f(x) + \epsilon * x\ \leq\ f(y) + \epsilon * y$. So this means that $f(x) - f(y) > 0$ and $f(x) - f(y) \leq \epsilon * (y - x)$. Let $\alpha = f(x) - f(y)$, then $\alpha > 0$, and $\alpha \leq \epsilon * (y - x)$. However we may choose $\epsilon = \frac{\alpha}{(y-x)*2} > 0$. Which give that $\alpha \leq \epsilon$*(y-x) $= \frac{\alpha}{2}$. Which is a contradiction. Hence $f(x)$ must be nondecreasing.

- **Now all we need to show is that $D^{+} f(x) \geq 0$ iff the other derivates are ≥ 0.**

  However we know that nondecreasing is equivalent to $D^{+} f(x) \geq 0$, and it follows immediately that $f(x+h) - f(x)$ and $f(x) - f(x-h)$ are each ≥ 0 for $h > 0$. However this means that the terms for which we are considering the limit of in each derivate must be always ≥ 0. So by definition all the derivates are ≥ 0 if $D^{+} f(x) \geq 0$.

  Now we need only look at the other direction. Clearly for both $D^{+} f(x) \geq D_{+} f(x)$ and $D^{-} f(x) \geq D_{+} f(x)$, so if respectively $D_{+} f(x)$ and $D_{-} f(x)$ are ≥ 0 than the other two will be. So in order to complete the other direction we only need that $D^{-} f(x) \geq 0$ implies that the function is nondecreasing. However if we replace $D^{+} f(x)$ with $D^{-} f(x)$ and the related definitions in the above two steps its easy to see that the remainder of the proof will preceed verbatim. Hence $D^{-} f(x) \geq 0$ implies that $f(x)$ is nondecreasing. This completes the proof. QED.