# Real Analysis - Math 630

## Homework Set #8 - Chapter 5

*by Bobby Rohde*
*11-03-00*

## Problem 7

- **a) Let $f$ be of bounded variation (BV) on $[a, b]$. Show that for each $c \in (a, b)$ the limit of $f(x)$ exists as $x \to c^-$ and also as $x \to c^+$. Prove that a monotone function (and hence a function of BV) can have only a countable number of discountinuities.**

- **Proof**

  - **Lemma: For a monotone BV function $g$ on $[a, b]$, the number of discontinuies such that $\forall \epsilon > 0, \left|g(x+\epsilon) - g(x-\epsilon)\right| \frac{>1}{n}, \forall \text{ n} \in \mathbb{N}$, is finite.**

    BWOC Suppose this number is infinite. Since it is a BV, we know that $g$ is strictly real valued. However we can start at $x = a$, and proceed towards $b$ such that each discontinuity we choose $\epsilon$ succifiently small that $\exists$ infintely many disconinuties between the current location and $b$. Thus we experience arbitrarily many increases of at least $\frac{1}{n}$, which since $g$ is monotonic implies that $g(b) = \infty$, which is a contradiction. Hence the number is finite.

  - **The number of discontinuities of $g$ as above is at most countable.**

    Let $A_n = \{x \,||g(x+\epsilon) - g(x-\epsilon)| \frac{>1}{n}, \forall \epsilon > 0\}$. It is clear that any discontinuity of a monotone function must be in some $A_n$ for $n$ sufficiently large. So the set of all discontinuities is $\bigcup_{n=1}^{\infty} A_n$, which must be countable since it is a countable union of finite sets. Thus the number of discontinuities is countable.

### The limit of $f(x)$ exists as $x \to c^-$ and $x \to c^+$

Consider a fixed c. Suppose $\exists\ \epsilon > 0$ such that $f$ over $(c - \epsilon, c)$, contains no discontinuties, then $f$ is continuous in that region and limit of $f(x)$ as $x \to c^-$exists.

Thus we may assume that $\forall\ \epsilon > 0\ \exists$ a discontuinity in $f$ over the region in question. We need to show that given $\gamma > 0$, $\exists\ \delta > 0$, such that $\forall\ x$ with $0 < c - x < \delta \Rightarrow |f(x) - f(c)| < \gamma$. However we know that $f$ is BV and thus the sum of two monotone functions $g$, $h$. Thus by the first lemma there exist only finitely many points with $|g(x) - g(c)| \frac{>1}{n}$. So we may take $\delta$ sufficiently small such that all points of differance > $\frac{\gamma}{2}$ are more than $\delta$ away. Similarly for $h$. Since they contribute with opposite sign, $|f(x) - f(c)| < \gamma$, were $\delta$ is the minimum of the two $\delta$'s needed.

Hence limit $x \to c^-$ exists. The proof however clearly proceeds likewise for $x \to c^+$. QED

## b) Construct a monotone function on $[a, b]$ which is discontinuous at each rational point.

## Construction

Let the sequence $< r_i >_{i=1}^{\infty}$, be an enumeration of the rationals in the interval $[0, 1]$. Define $f_r(x) = \begin{cases} r, & x \geq r \\ 0, & x < r \end{cases}$.

Let $f(x) = \sum_{i=1}^{\infty} f_{r_i}(x) * \frac{1}{2^n}$. Claim that $f$ is such a function as required.

Clearly $\sum_{i=1}^{\infty} f_{r_i}(x) * \frac{1}{2^n} \leq \sum_{i=1}^{\infty} \frac{1}{2^n} = 1$. Thus since the lefthand side is a monotonically increasing bounded series it must converge to some number less than infinity. Furthermore it should be clear that $\forall\ x, y \in [0, 1]$ with $x < y$, we have that $\exists$ some $q \in (x, y) \cap \mathbb{Q}$, since rationals are dense. So $f(y) - f(x) \geq f_q(y) * \frac{1}{2^N}$, for some $N$ since all terms for which $f_{r_i}(x) \neq 0$ yields $f_{r_i}(x) = f_{r_i}(y)$, and $f_q(x) = 0 \neq f_q(y)$. Thus $f(x)$ is monotonicly increasing.

Furthermore $\forall\ p \in \mathbb{Q}$, and $\forall\ h > 0$. $f(p+h) - f(p-h) \geq f_p(p+h) * \frac{1}{2^N}$, using the above property. However since $p \in \mathbb{Q}$ implies that $f_p(p+h) = f_p(p) = p$. Thus $\forall\ h > 0$, $f(p+h) - f(p-h) \geq \frac{p}{2^N} > 0$. Which implies that $f(p+h) - f(p-h) \geq \frac{p}{2^N}$, so limit h → 0, $\lim_{x \to p^+} f(x) - \lim_{x \to p^-} f(x) \geq \frac{p}{2^N}$. Hence $f(x)$ is discontinuous at $p$ since the limits can not be equal.

Thus $f$ satisfies all the conditions desired. QED.

## Problem 8

### a) Show that if $a \le c \le b$, then $T_a^b = T_a^c + T_c^b$ and that hence $T_a^c \le T_a^b$.

### Proof

- $T_a^b \le T_a^c + T_c^b$

By definition $T_a^b = \sup t$, where sup is taken over all possible partitions of $[a, b]$. However $t = \sum_{i=1}^{k} |f(x_i) - f(x_{i-1})|$. Then either $c = x_i$, for some $i$, or $c \in (x_i, x_{i+1})$, for some $i$.

If $c = x_i$, then clearly we may split the partition over $[a, b]$ into two partitions, one over $[a, c]$ and the other over $[c, b]$, such that $t_a^b$ is equal to $t_a^c + t_c^b$, for those partitions.

If $c \in (x_i, x_{i+1})$ then consider the partition $a = x_0 < x_1 < \ldots < x_i < c < x_{i+1} < \ldots < x_k = b$. In this partition $t_2$ is $\ge$ to the original $t \equiv t_1$, since $|f(x_{i+1}) - f(x_i)| = |f(x_{i+1}) - f(c) + f(c) - f(x_i)| \le |f(x_{i+1}) - f(c)| + |f(c) - f(x_i)|$, by triangle inequality. If we break the new partition as above we have two partitions about $[a, c]$ and $[c, b]$, respectively with $t$ taken over these partitions summed = $t_2 \ge t_1$. Hence $T_a^c + T_c^b \ge T_a^b$.

- $T_a^b \ge T_a^c + T_c^b$

Every partition $p_1$, $p_2$ over $[a, c]$ and $[c, b]$ respectively can be used to form a partition over $[a, b]$ by joining the two at $c$. However $T_a^b = \sup t$ over all possible partitions, so this includes the partition formed by the union of $p_1$ and $p_2$ and hence $T_a^b \ge T_a^c + T_c^b$.

Thus $T_a^b = T_a^c + T_c^b$. Showing that $T_a^c \le T_a^b$.

### b) Show that $T_a^b(f + g) \le T_a^b(f) + T_a^b(g)$, and $T_a^b(c * f) = |c| * T_a^b(f)$.

### Proof

- $T_a^b(f + g) \le T_a^b(f) + T_a^b(g)$

$T_a^b(f + g) = \sup t(f + g)$, where $t(f + g) = \sum_{i=1}^{k} |(f + g)(x_i) - (f + g)(x_{i-1})| = \sum_{i=1}^{k} |f(x_i) + g(x_i) - f(x_{i-1}) - g(x_{i-1})| \le \sum_{i=1}^{k} (|f(x_i) - f(x_{i-1})| + |g(x_i) - g(x_{i-1})|)$ by the triangle inequality, however this last term is just $t(f) + t(g)$.

Thus $\sup t(f + g) \le \sup (t(f) + t(g))$, hence $T_a^b(f + g) \le T_a^b(f) + T_a^b(g)$. QED

- **$T_a^b(c*f) = |c| * T_a^b(f)$**

  $T_a^b(c*f) = \sup t(c*f)$, where $t(c*f) = \sum_{i=1}^k |c*f(x_i) - c*f(x_{i-1})| = \sum_{i=1}^k |c| *| f(x_i) - f(x_{i-1})| = \left|c\right| * \sum_{i=1}^k \left| f(x_i) - f(x_{i-1}) \right| = |c| * t(f)$.

  Thus $\sup t(c*f) = \sup |c| * t(f) = |c| * \sup t(f)$, hence $T_a^b(c*f) = |c| * T_a^b(f)$. QED

---

## Problem 10

- **a) Let $f$ be defined by $f(0) = 0$ and $f(x) = x^2 \sin(\frac{1}{x^2})$, for $x \neq 0$. Is $f$ of BV on $[-1, 1]$?**

- **Answer**

  No. Over any interval $(a, b)$, the diffence from $x^2 \sin(\frac{1}{x^2})$, will be $= a^2 \sin(\frac{1}{a^2}) - b^2 \sin(\frac{1}{b^2})$. Suppose a is a minimum, thus $\frac{1}{a^2} = n * \pi - \frac{\pi}{2}$, for some integer $n$. Thus $\exists$ an adjacent maximum at $b$ such that $\frac{1}{b^2} = n * \pi + \frac{\pi}{2}$. So $a^2 \sin(\frac{1}{a^2}) - b^2 \sin(\frac{1}{b^2}) = \frac{1}{n*\pi - \frac{\pi}{2}} (-1) - \frac{1}{n*\pi + \frac{\pi}{2}} * 1 = \frac{-2}{(2*n-1)*\pi} + \frac{-2}{(2*n+1)*\pi} = \frac{-8*n}{(4 n^2 - 1)*\pi}$. However as the number of partitions gets arbitrarily large we may add arbitrarily many terms of this form with $n$ increasing. So $T_{-1}^0(f) \geq \sum_{n=1}^\infty \left| \frac{-8*n}{(4 n^2 - 1)*\pi} \right| \geq \sum_{n=1}^\infty 2 * \frac{1}{n*\pi} = \infty$, since the last series is a constant times the harmonic series. Therefore $f$ is not a BV.

## ■ b) Let $g$ be defined by $g(0) = 0$ and $g(x) = x^2 \sin(\frac{1}{x})$, for $x \neq 0$. Is $g$ of BV on $[-1, 1]$?

## ■ Answer

Yes. Since $|g(x)|$ is symmetric in $x$ we know that it is suffiecent to show that $g(x)$ is BV on $[-1, 0]$.

Consider a single transition for a minimum to an adjacent maximum. It is clear that it will proceed monotonically in this region. On inspection it is clear that $t$ over a monotonic region will be independant of the choice of partition and have the value equal to the differance of the end points. (Either $p$ or $n$ is 0).

From problem 8a) and an obvious induction it is clear that $T_a^b$ may be broken into finitely many pieces. We know that minima and maxima occur at $\sin(\frac{1}{x}) = \pm 1 \Rightarrow x = \frac{1}{n*\pi \pm \frac{\pi}{2}}$. It is obvious that $T_{-1}^{\frac{-2}{\pi}} < \infty$, purely by inspection and our knowledge of basic calculus. Proceed by breaking $T_a^b$ into $m$ pieces such that the first piece is $T_{-1}^{\frac{-2}{\pi}}$, and the proceeding $m$-2 pieces follow for decreasing values of $n$, and the last piece represents the tail going to 0.

Now we need to evaluate $T$ over one these pieces that have been set up to run from a local max to a local min or vice versa. So the points in question have the form $\left[\frac{1}{n*\pi+\frac{\pi}{2}}, \frac{1}{(n-1)*\pi+\frac{\pi}{2}}\right]$. So $\left|\left(\frac{1}{n*\pi+\frac{\pi}{2}}\right)^2 * \sin(n*\pi + \frac{\pi}{2}) - \left(\frac{1}{(n-1)*\pi+\frac{\pi}{2}}\right)^2 * \sin((n-1)*\pi + \frac{\pi}{2})\right| = \left|\left(\frac{1}{n*\pi+\frac{\pi}{2}}\right)^2 + \left(\frac{1}{(n-1)*\pi+\frac{\pi}{2}}\right)^2\right| = \left|\frac{2*(n*\pi)^2+2*(\frac{\pi}{2})^2}{(n*\pi+\frac{\pi}{2})^2\,((n-1)*\pi+\frac{\pi}{2})^2}\right| < \left|\frac{2*(n^2+\frac{1}{4})}{(n-1)^2*(n-2)^2}\right| < \left|\frac{2*(n^2+\frac{1}{4})}{(n-2)^4}\right|$, note that the inequalities depend on the fact that $n < 0$.

Now we want to consider the total $T_{-1}^{0} \leq T_{-1}^{\frac{-2}{\pi}} + \sum_{n=-1}^{-(m-2)} \left|\frac{2*(n^2+\frac{1}{4})}{(n-2)^4}\right| + T_{\text{tail}}^{0}$, where $T_{\text{tail}}^{0}$ is the part not covered in $m$ steps. Taking limit $m \to \infty$, the tail contribution must go to 0. So we only need that the $\sum_{n=-1}^{-\infty} \left|\frac{2*(n^2+\frac{1}{4})}{(n-2)^4}\right| < \infty$. However this may be rewritten as $\sum_{n=1}^{\infty} \left|\frac{2*(n^2+\frac{1}{4})}{(2-n)^4}\right|$. However this is bounded above by $\frac{3}{n^2}$ for $n$ large, which is a convergent sequence. Thus our sum must converge. Hence $T_{-1}^{0} < \infty$. So $T_{-1}^{1} < \infty \Rightarrow x^2 \sin(\frac{1}{x})$ is BV. QED.

## Problem 14

### a) Show that the sum and differance of two AC functions are also AC.

### Proof

Let $f$ and $g$ be AC. Let $\epsilon > 0$. Then $\exists\ \delta_1, \delta_2$ such that $\sum_{i=1}^{n} |f(x_i') - f(x_i)| \frac{<\epsilon}{2}$ and $\sum_{i=1}^{n} | g(x_i') - g(x_i)| \frac{<\epsilon}{2} \forall$ finite collection of intervals $\{(x_i, x_i')\}$, such that $\sum_{i=1}^{n} |x_i' - x_i| < \delta = \min\{\delta_1, \delta_2\}$.

However $\sum_{i=1}^{n} | f(x_i') + g(x_i') - f(x_i) - g(x_i)| \leq \sum_{i=1}^{n} |f(x_i') - f(x_i)| + | g(x_i') - g(x_i)|$, by triangle inequlaity. Further $\sum_{i=1}^{n} |f(x_i') - f(x_i)| + | g(x_i') - g(x_i)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$.

Hence we may choose $\delta = \min\{\delta_1, \delta_2\}$ in order to satisfy the condition and show that $f + g$ is AC.

Since $|f(x_i') - f(x_i)| = |-(f(x_i') - f(x_i))| = |(-f(x_i')) - (-f(x_i))|$, we know that *f* is AC implies that -*f* is AC. So $f - g$ must also be AC.

### b) Show that the product of two AC functions is AC

### Proof

#### Lemma: *f* is AC implies that $(f)^2$ is AC.

Let *f* be AC with respect to $[a, b]$. Then we know that *f* is BV, and hence that since $[a, b]$ is a closed interval, that $|f|$ is bounded. Let $M > 0$ be a bound on $|f|$. Let $\epsilon > 0$. Since *f* is AC, $\exists\ \delta > 0$ such that $\sum_{i=1}^{n} |f(x_i') - f(x_i)| \frac{<\epsilon}{2M}$, $\forall$ finite collection of intervals $\{(x_i, x_i')\}$, such that $\sum_{i=1}^{n} | x_i' - x_i| < \delta$.

Consider $\sum_{i=1}^{n} |(f(x_i'))^2 - (f(x_i))^2| = \sum_{i=1}^{n} |f(x_i') - f(x_i)| * | f(x_i') + f(x_i)| \leq \sum_{i=1}^{n} |f(x_i') - f(x_i)| * (|f(x_i')| + |f(x_i)|) \leq \sum_{i=1}^{n} |f(x_i') - f(x_i)| * 2M < \frac{\epsilon}{2M} * 2M = \epsilon$.

Thus $(f)^2$ is AC. Concluding the Lemma.

- **If $f$, $g$ are AC over $[a, b]$ then $f * g$ is AC.**

  $f * g = \frac{(f+g)^2-(f)^2-(g)^2}{2}$, but that both sums, differances and squares of AC functions are AC, so we only need that constant multiples of AC functions are AC. But for $c * f$, where $f$ is AC it follows immediately that for any $\epsilon > 0$ we may choose $f$ constrained by $\frac{\epsilon}{c}$, and the $c$ will carry through to make $c * f$, AC. (Note $c = 0$ is obviously true.)

  Hence $f * g$ can be gotten as a sequence of operations that perserve AC, so it is also AC. QED.

## c) If $f$ is AC on $[a, b]$ and if $f$ is never zero there, then the function $g = \frac{1}{f}$ is also AC on $[a, b]$.

## Proof

Since $f$ is never 0 over a closed interval, we know that $\exists\ m > 0$ such that $|f| > m$ everywhere. Let $\epsilon > 0$. Since $f$ is AC, $\exists\ \delta > 0$, such that $\sum_{i=1}^{n} |f(x_i') - f(x_i)| < \epsilon * m^2$.

Consider $\sum_{i=1}^{n} \left| \frac{1}{f(x_i')} - \frac{1}{f(x_i)} \right| = \sum_{i=1}^{n} \left| \frac{f(x_i)-f(x_i')}{f(x_i')*f(x_i)} \right| \leq \sum_{i=1}^{n} \left| \frac{f(x_i)-f(x_i')}{m^2} \right| < \frac{\epsilon * m^2}{m^2} = \epsilon$.

Thus $g = \frac{1}{f}$ is AC. QED.