Real Analysis - Math 630

Homework Set #9 - Chapters 5 and 6

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Problem 12

- **Let f be absolutely continuous (AC) in the interval $[\epsilon, 1]$, $\forall\, \epsilon > 0$. Does the continuity of f at 0 imply that f is AC on [0, 1]? What if f is also a bounded variation (BV) on [0, 1]?**

- **Proof**

  - **Does the continuity of f at 0 imply that f is AC on [0, 1]?**

    No. Consider for example the function $f = x^2 * \sin(\frac{1}{x^2})$. Over the interval $[\epsilon, 1]$, we know that f is expressable as $f(x) = \int_\epsilon^x f'(t)\, dt + f(\epsilon)$, where $f' = 2\,x * \sin(\frac{1}{x^2}) - \frac{2}{x}\cos(\frac{1}{x^2})$, which is well-defined over $[\epsilon, 1]$ as is $f(\epsilon)$. Thus by Theorem 14, f is AC over $[\epsilon, 1]$.

    However we know from the result of Problem 10a, that f is not BV on [0, 1] and hence by the contrapositive of Lemma 11, we know that f is not AC on [0, 1].

  - **What if f is also a bounded variation (BV) on [0, 1]?**

    Yes, this suffices. Consider an f that is BV on [0, 1], AC on $[\epsilon, 1]$ $\forall\, \epsilon > 0$ and continuous at 0. We essentially need to show that $\sum_{i=1}^n |\, f(x_i{}') - f(x_i)\,| \to 0$ over disjoint intervals $(x_i{}', x_i) \subset [0, \epsilon]$ as $\epsilon \to 0$, as this would ensure that any additional intervals added to to an existing AC description could be bounded by choosing sufficiently smaller δ.

    Given that f is BV we then have that over any partition $\sum_{i=1}^n |\, f(x_i) - f(x_{i-1})\,| < \infty$. Also since it's continuous at $x = 0$, we know that $\lim_{x\to 0} |f(x) - f(0)| = 0$. Since by Problem 8 we know that $T_a^b = T_a^c + T_c^b$, with $c \in (a, b)$, this says that, $T_1^0 = T_1^\epsilon + T_\epsilon^0$. And the above statement gives that $T_\epsilon^0 \to 0$, as $\epsilon \to 0$. Hence $\sum_{i=1}^n |\, f(x_i) - f(x_{i-1})\,|$, over inteverals $\subset [0, \epsilon]$ goes to 0. Hence giving that $\sum_{i=1}^n |\, f(x_i{}') - f(x_i)\,| \to 0$ as desired. So f is AC. QED.

Problem 20

- **A function f is said to satisfy a Lipschitz condition (LC) on an interval if $\exists$ a constant M such that $|f(x)-f(y)| \le M * |x-y|$, $\forall$ x and y in the interval.**

- **a) Show that a function satisfying a LC is AC.**

- **Proof**

Let f be an LC function with constant M. If $M=0$ then f is constant so f is trivially AC. Thus we may assume $M>0$. Then $\forall$ finite collections of non-overlapping intervals $\{(x_i', x_i)\}$ we know that $\sum_{i=1}^n |x_i' - x_i| \ge \sum_{i=1}^n \frac{|f(x_i')-f(x_i)|}{M}$, by the LC on each term of the sum. Given $\epsilon > 0$. If we take $\delta = \frac{\epsilon}{M}$, then $\delta > \sum_{i=1}^n |x_i' - x_i| \Rightarrow \frac{\epsilon}{M} = \delta > \sum_{i=1}^n \frac{|f(x_i')-f(x_i)|}{M} \Rightarrow \epsilon > \sum_{i=1}^n |f(x_i') - f(x_i)|$. QED.

- **b) Show that an AC function f satisfies a LC iff $|f'|$ is bounded.**

- **Proof**

Suppose $f'(x)$ exists. Then we know that $f'(x) = D^+ f(x) = \overline{\lim}_{h\to 0^+} \frac{f(x+h)-f(x)}{h} = \overline{\lim}_{y\to x^+} \frac{f(y)-f(x)}{y-x}$. Hence $|f'| = \overline{\lim}_{y\to x^+} \frac{|f(y)-f(x)|}{|y-x|}$.

If f is LC then $|f(x)-f(y)| \le M * |x-y| \Rightarrow |f'| = \overline{\lim}_{y\to x^+} \frac{|f(y)-f(x)|}{|y-x|} \le \overline{\lim}_{y\to x^+} \frac{M*|y-x|}{|y-x|} = M$. $\therefore$ if f is LC then $|f'| \le M$.

If $|f'| \le M$, then $M \ge |f'(x)| = \overline{\lim}_{y\to x^+} \frac{|f(y)-f(x)|}{|y-x|} = \lim_{y\to x^+} \frac{|f(y)-f(x)|}{|y-x|}$, since $f'(x)$ exists. $\Rightarrow$ $\forall$ $\epsilon > 0$, $\exists$ $\delta > 0$, such that $0 < |y-x| < \delta \Rightarrow \left| \frac{|f(y)-f(x)|}{|y-x|} - \left| f'(x) \right| \right| < \epsilon$, $\Rightarrow$ $\left| |f(y)-f(x)| - |f'(x)| * |y-x| \right| < \epsilon * |y-x|$. Applyng the triangle inequality to the left hand sided and rearranging terms we have that $|f(y)-f(x)| < |y-x| * (|f'(x)| + \epsilon)$. $\Rightarrow$ $|f(y)-f(x)| < |y-x| * (M+\epsilon)$.

This expression is true for all y, such that $0 < |y-x| < \delta$, however our choice of x is arbitrary so that there must exist a δ associated with any x which makes the above true. Further since $|y-x| \le |y-a| + |a-x|$, $\forall$ $a \in (x, y)$ we know that $|f(y)-f(x)| < |y-x| * (M+\epsilon)$ must hold for all x, y.

Finally we may take the limit as $\epsilon \to 0$. This gives that $|f(y)-f(x)| \le |y-x| * M$. QED

Problem 1

- **Show that $\| f + g \|_\infty \leq \| f \|_\infty + \| g \|_\infty$**

- **Proof**

By definition $\| f \|_\infty =$ ess sup $|f(t)| = \inf\{M : m\{t : |f(t)| > M\} = 0\}$, $\| g \|_\infty = \inf\{N : m\{t : |g(t)| > N\} = 0\}$, and $\| f + g \|_\infty = \inf\{L : m\{t : |f(t) + g(t)| > L\} = 0\}$.

This last gives $\inf\{L : m\{t : |f(t) + g(t)| > L\} = 0\} \leq \inf\{L : m\{t : |f(t)| + |g(t)| > L\} = 0\} \leq \inf\{M + N : m(\{t : |f(t)| > M\} \bigcup \{t : |g(t)| > N\}) = 0\}$, since this last union clearly includes all points with $|f(t)| + |g(t)| > M + N$, as well as many others as well.

However, $\inf\{M + N : m(\{t : |f(t)| > M\} \bigcup \{t : |g(t)| > N\}) = 0\} \leq \inf\{M + N : m\{t : |f(t)| > M\} + m\{t : |g(t)| > N\}) = 0\}$, by subadditivity. Further though we may require that each set have measure zero since measure is non-negative. Finally this allows us to break it into two independant infimums yielding. $\inf\{M : m\{t : |f(t)| > M\} = 0\} + \inf\{N : m\{t : |g(t)| > N\} = 0\}$ which gives us that $\| f + g \|_\infty \leq \| f \|_\infty + \| g \|_\infty$. QED.

Problem 2

- **Let f be a bounded measurable function on [0, 1]. Then $\lim_{p\to\infty} \| f \|_p = \| f \|_\infty$**

- **Proof**

$\| f \|_p = \left(\int_0^1 (|f|)^p\right)^{\frac{1}{p}}$. Clearly if we replace $|f|$ with $M =$ ess sup f we have $\left(\int_0^1 (|f|)^p\right)^{\frac{1}{p}} \leq \left(\int_0^1 M^p\right)^{\frac{1}{p}} = M = \| f \|_\infty$. Hence $\| f \|_p \leq \| f \|_\infty$, $\forall\ p$. Take $\epsilon > 0$. Define $E = \{x \in [0, 1] : |f| > M - \epsilon\}$ and $D = [0, 1] \sim E$. $\left(\int_0^1 (|f|)^p\right)^{\frac{1}{p}} = \left(\int_E (|f|)^p + \int_D (|f|)^p\right)^{\frac{1}{p}} \geq \left((M - \epsilon)^p * mE + \int_D (|f|)^p\right)^{\frac{1}{p}} \geq (M - \epsilon) * mE^{\frac{1}{p}}$. We know that $mE > 0$ since M is the ess sup of f. If $mE = 0$ then there would be a lower essential sup $\leq M - \epsilon$. However $\forall\ \alpha > 0$, $\alpha^{\frac{1}{p}} \to 1$ as $p \to \infty$. So in the limit as $p \to \infty$, $(M - \epsilon) * mE^{\frac{1}{p}} \to (M - \epsilon)$. However since ϵ can be taken arbitratily small. We thus have that $\lim_{p\to\infty} \| f \|_p \geq M = \| f \|_\infty$. Thus since each term in the limit is less than M. It must be the case that $\lim_{p\to\infty} \| f \|_p = \| f \|_\infty$ QED.

Problem 3

- **Prove that $\| f + g \|_1 \le \| f \|_1 + \| g \|_1$**

- **Proof**

$\| f + g \|_1 = \int_0^1 | f + g | \le \int_0^1 (|f| + |g|) = \int_0^1 \left| f \right| + \int_0^1 |g| = \| f \|_1 + \| g \|_1$. QED.

Problem 4

- **If $f \in L^1$ and $g \in L^\infty$, then $\int |f * g| \le \| f \|_1 * \| g \|_\infty$**

- **Proof**

$\int |f * g| = \int (|f| * |g|) \le \int (|f| * \text{ess sup } |g|) = \text{ess sup } |g| * \int |f| = \| f \|_1 * \| g \|_\infty$. QED.

Extra Credit - Problem 5.21

- **Let g be a monotone increasing AC function on $[a, b]$ with $g(a) = c,\ g(b) = d$.**

- **a) Show that for any open set $O \subset [c, d]$, $mO = \int_{g^{-1}[O]} g'(x)\, dx$**

- **Proof**

Since O is an open set we know that it is expressible as the union of a finite collection of open intervals $\{(c_i, d_i)\}$. Furthermore since g is AC that implies it is continuous, and since it is monotone increasing we know from elementary analysis that the pre-image of an open interval must be an open interval (possiblity degenerate or empty), with endpoints $g^{-1}(c_i), g^{-1}(d_i)$. Thus $\int_{g^{-1}[O]} g'(x)\, dx = \sum_{i=1}^n \int_{g^{-1}(c_i)}^{g^{-1}(d_i)} g'(x)\, dx$. However since g is AC, we further know that $g(x) = \int_a^x g'(t)\, dt + g(a)$.

Hence $\sum_{i=1}^n \int_{g^{-1}(c_i)}^{g^{-1}(d_i)} g'(x)\, dx = \sum_{i=1}^n (g(g^{-1}(d_i)) - g(g^{-1}(c_i)))$. By definition of inverse we have that this is equal to $\sum_{i=1}^n (d_i - c_i) = \sum_{i=1}^n l(c_i, d_i) = mO$. QED.

■ b) Let $H = \{x : g'(x) \neq 0\}$. If E is a subset of $[c, d]$ with $mE = 0$, then $g^{-1}(E) \bigcap H$ has measure zero.

■ Proof

BWOC Suppose $g^{-1}(E) \bigcap H$ has measure > 0. Then $\int_{g^{-1}(E)\bigcap H} |g'| > 0$. Since the integration is preformed over a set of measure > 0 where $|g'|$ is also > 0 everywhere. Since g is monotone increasing we may neglect the absolute value since the derivate must be ≥ 0.

Further we know that $\int_{g^{-1}(E)\bigcap H} g' = \int_{g^{-1}(E)} g'$, since when integrating over the complement of H we have that $g' = 0$. So since $g^{-1}(E) \bigcap H$ has measure > 0, $g^{-1}(E)$ has measure > 0, and thus ∃ some open interval $(\alpha, \beta) \subset g^{-1}(E)$.

$\int_\alpha^\beta g' \, dx = g(\alpha) - g(\beta)$, but $g[(\alpha, \beta)] \subset g(g^{-1}(E)) = E \Rightarrow$ that $g[(\alpha, \beta)] = \{\gamma\} \Rightarrow g(\alpha) - g(\beta) = 0$. Which contradicts the fact that $\int_{g^{-1}(E)\bigcap H} |g'| > 0$. Hence we have that $g^{-1}(E) \bigcap H$ has measure zero. QED.

■ c) If E is a measurable subset of $[c, d]$, then $F = g^{-1}[E] \bigcap H$ is measurable and $mE = \int_F g' = \int_a^b \chi_E(g(x)) * g'(x) \, dx$

■ Proof

$\int_F g' = \int_{g^{-1}[E]\bigcap H} g'$. Clearly we may neglect the H, since if we are not in H then $g' = 0$ and contributes nothing to the integral. So $\int_F g' = \int_{g^{-1}[E]} g'$. We know that ∃ an open set O with $O \subset E$ and $mE = mO$. By part b) we may neglect the points in $E \sim O$ since they are a set of measure 0. So considering $\int_{g^{-1}[O]} g'$ and invoking part a) we get $\int_F g' = mO = mE$.

But also $\int_{g^{-1}[O]} g' = \int_a^b g'(x) * \chi_{g^{-1}[O]}(x) \, dx$. However clearly $x \in g^{-1}[O]$ iff $g(x) \in O \Rightarrow \chi_{g^{-1}[O]}(x) = \chi_O(g(x))$. $\Rightarrow \int_a^b g'(x) * \chi_{g^{-1}[O]}(x) \, dx = \int_a^b g'(x) * \chi_O(g(x)) \, dx$. Finally since adding a set of measure 0 does not change the value of the integral we may replace O with E. Leaving $\int_a^b g'(x) * \chi_E(g(x)) \, dx = \int_F g' = mE$. QED

■ d) If f is a nonnegative measurable function on $[c, d]$, then $(f \circ g)(g')$ is measurable on $[a, b]$ and $\int_a^b f(y) \, dy = \int_a^b f(g(x)) * g'(x) \, dx$.

■ Proof

I'm tired. Good night.